Slivers

Topics: classifying tetrahedra, periodic meshes, weighted points, weighted Delaunay complex, pumping lemma, sliver theorem.

Classifying tetrahedra. Recall the ratio property from last section, which for every tetrahedron bounds the ratio R/L from above by a constant ϱ_0. Here R is the circumradius and L is the length of the shortest edge of the tetrahedron. This property eliminates all badly shaped triangles and most but not all badly shaped tetrahedra. Tetrahedra that have all four vertices close to a line are shown in Figure 14, and they are prohibited by the ratio property because they all contain at least one badly shaped triangle. Tetrahedra that

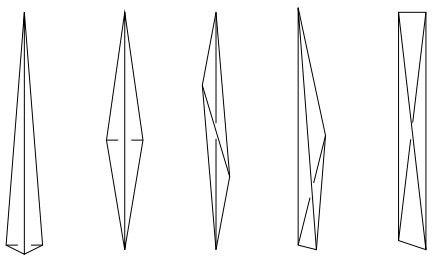

Figure 14: Badly shaped skinny tetrahedra.

have all four vertices close to a plane (but not to a line) are shown in Figure 15. The left two tetrahedra both have at least one badly shaped triangle. The cap has no badly shaped triangle, but its circumsphere is large relative to the length of its edges and it is also prohibited by the ratio property. However, slivers can have circumspheres with radius almost as small as $L/\sqrt{2}$ and are thus in the position to satisfy the ratio property unless $\varrho_0 < 1/\sqrt{2}$. Experiments with 3-dimensional Delaunay complexes show that a positive fraction of their tetrahedra tend to be slivers, even if points are added to guarantee the ratio property for a small but not very small value of ϱ_0.

Periodic meshes. The goal of this section is to explain that if the Delaunay complex of a finite set $S \subseteq \mathbb{R}^3$ has the ratio property then there is another complex K

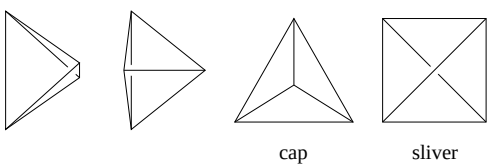

Figure 15: Badly shaped flat tetrahedra.

with the same vertex set and underlying space that has the ratio property and no sliver. This cannot be true in general, for if S contains only the four vertices of a sliver then there is no other tetrahedron we can form. Similar counterexamples based on constraints at the boundary exist for larger point sets. We have to somehow finesse the boundary problem. Instead of constraining angles between segments and facets of an input polyhedron, as we did in Section 19, we assume that S is periodic and we construct a periodic 3-complex that triangulates the entire $\mathbb{R}^3$. Specifically, $S_0 \subseteq [0,1)^3$ is finite and

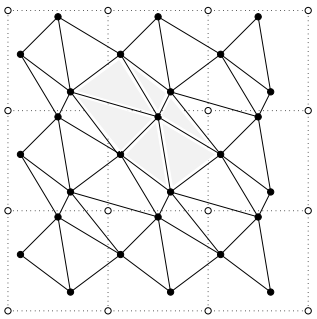

Figure 16: Periodic tiling of the plane. The shaded triangles form a fundamental domain whose shifted copies tile the entire plane.

$S = S_0 + \mathbb{Z}^3$, where $\mathbb{Z}^3$ is the three-dimensional integer grid. A 3-complex D with vertex set S and $|D| = \mathbb{R}^3$ is *periodic* if $\sigma \in D$ implies $\sigma + \mathbb{Z}^3 \subseteq D$. Figure 16 illustrates this idea in two dimensions.

Weighted points. The sliver-free 3-complex K will be a weighted Delaunay complex. To prepare the definition of such complexes we introduce points with real weights. It is convenient to write the weight of a point as the square of a non-negative real or a non-negative multiple of the imaginary unit. We think of the weighted point $\hat{p} = (p, P^2) \in \mathbb{R}^3 \times \mathbb{R}$ as the sphere with center $p \in \mathbb{R}^3$ and radius P. The *power* or *weighted distance* of a point $x \in \mathbb{R}^3$ from $\hat{p}$ is

$$\pi_{\hat{p}}(x) \quad = \quad \|x - p\|^2 - P^2.$$

For positive P^2 and x outside the sphere this is the square length of a tangent line segment that connects x with a point on the sphere, see Figure 17. In Section 2

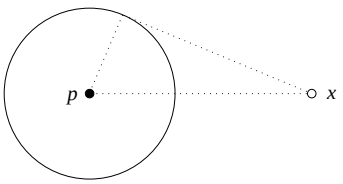

Figure 17: The segment px, the tangent segment from x to the circle, and the connecting radius form a right triangle.

we have seen that the set of points with equal weighted distance from two circles is a line. Similarly, the set of points with equal weighted distance from two spheres in $\mathbb{R}^3$ is a plane. If the two spheres intersect then the plane passes through the intersection circle, and if the two spheres are disjoint and lie side by side then the plane separates the two spheres.

Two weighted points $\hat{p}$ and $\hat{q} = (q, Q^2)$ are *orthogonal* if $\|p - q\|^2 = P^2 + Q^2$. In the real case the two spheres intersect in a circle, and for every point on that circle the tangent planes to the two spheres form a right angle. We call $\hat{p}$ and $\hat{q}$ *further than orthogonal* if $\|p - q\|^2 > P^2 + Q^2$.

Weighted Delaunay complex. Let $\hat{S}$ be a set of weighted points in $\mathbb{R}^3$. The weighted distance can be used to construct the (weighted) Voronoi regions of the $\hat{p} \in \hat{S}$. The *(weighted) Delaunay complex* is the dual of the set of Voronoi regions, as usual. Instead of going through the technical formalism of the construction, which is pretty much the same as for unweighted points, we illustrate the concept in Figure 18. For unweighted points, a tetrahedron belongs to the Delaunay complex iff the circumsphere passing through the four vertices is empty. For weighted points the circumsphere is replaced by the *orthosphere*, which is the unique sphere

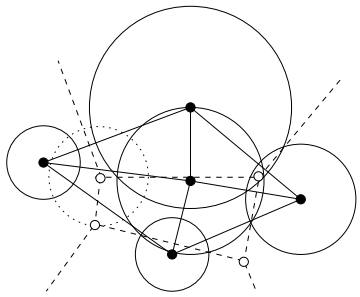

Figure 18: Voronoi regions and the dual Delaunay triangulation of five weighted points in the plane. Each Voronoi vertex is the center of a circle orthogonal to the generating circles of the regions that meet at that vertex. Only one such circle is shown.

orthogonal to all four spheres whose centers are the vertices of the tetrahedron. Its center is the corner shared by the four (weighted) Voronoi regions, and its weight is the common weighted distance of that corner from the four spheres, see Figure 18.

CLAIM 8. A tetrahedron belongs to the weighted Delaunay complex iff the orthosphere of the four weighted points is further than orthogonal from all other weighted points in the set.

Voronoi regions of weighted points can be empty. As a consequence, the weighted Delaunay complex may use only a subset of the points in S as vertices. However, if the spheres in $\hat{S}$ all have non-negative radius and are pairwise disjoint and lie side by side then every weighted point in $\hat{S}$ has a non-empty Voronoi region and every point in S is a vertex of the weighted Delaunay complex of $\hat{S}$. We restrict outselves to such sets $\hat{S}$.

Pumping lemma. A crucial observation in the elimination of slivers is the instability of their circumspheres or orthospheres. With a small weight change we can increase the size of the orthosphere dramatically. This is useful because a large orthosphere is likely to violate Claim 8, and as a consequence the sliver excuses itself from the weighted Delaunay complex.

We quantify the impact of pumping up the weight of a sliver vertex. Let $\hat{a}, \hat{b}, \hat{c}, \hat{p}$ be weighted points, and H the distance of the center of the orthosphere from the plane passing through abc, see Figure 19. Consider varying the weight P^2 of $\hat{p}$ and think of H as a function $\mathbb{R} \to \mathbb{R}$ of that one variable. Let δ be the distance of p from the plane of abc.

LEMMA 9. $H(P^2) = H(0) + \frac{P^2}{2\delta}$.

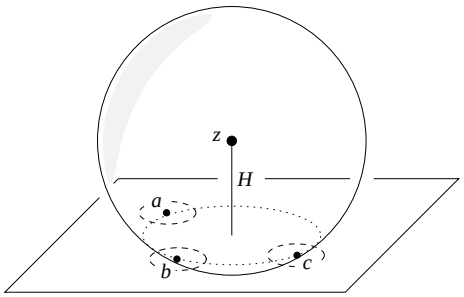

Figure 19: The orthosphere of weighted points $\hat{a}, \hat{b}, \hat{c}$ and another weighted point $\hat{p}$ not shown in the figure.

PROOF. As P^2 varies, the center of the orthosphere moves on a line orthogonal to the plane of abc. Let λ be the distance of p from that line, see Figure 20. Then $Z^2 + P^2 = (H(P^2) - \delta)^2 + \lambda^2$. The square radius of

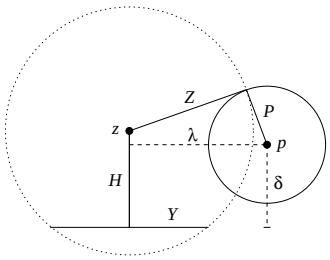

Figure 20: The orthocenter z moves downwards as the weight of p increases.

the orthosphere is $Z^2 = H(P^2)^2 + Y^2$, where Y^2 is the square radius of the orthocircle of $\hat{a}, \hat{b}, \hat{c}$ in the plane of abc. Hence

$$\begin{aligned} H(P^2)^2 &= Z^2 - Y^2 \\ &= (H(P^2) - \delta)^2 + \lambda^2 - P^2 - Y^2, \end{aligned}$$

and after cancelling $H(P^2)^2$ we get

$$H(P^2) = \frac{\delta^2 + \lambda^2 - Y^2}{2\delta} - \frac{P^2}{2\delta}.$$

The first term on the right side is $H(0)$. □

The term $P^2/2\delta$ is the displacement of the orthocenter if we change the weight of $\hat{p}$ from 0 to P^2. For slivers the value of δ is small so the displacement is large.

Sliver theorem. We state the main result without proof, which is fairly technical and involved. We need a measure that distinguishes slivers from other tetrahedra, and we use $\xi = V/L^3$, where V is the volume and L is the length of the shortest edge. Tetrahedra with small values of ξ can be slivers or other badly shaped tetrahedra shown in Figures 14 and 15, but if we assume $R/L \leq \varrho_0$ then a small value of ξ identifies a sliver. Let S be a periodic set of points in $\mathbb{R}^3$ and assume the Delaunay complex of S has ratio property for some constant ϱ_0.

THEOREM 10. There are constant ϱ_1 and ξ_0 and a weight assignment $\hat{S}$ such the weighted Delaunay complex of $\hat{S}$ has the ratio property for ϱ_1 and $\xi > \xi_0$ for all its tetrahedra.

Bibliographic notes. The occurrence of slivers as a menace in 3-dimensional Delaunay complexes was reported by Cavendish, Field and Frey [1]. The only theoretical results prior to the work described in this section are contained in a three page correspondence by Chew [3]. The material of this section is taken from Cheng et al. [2], which contains the technical details needed to prove Theorem 10. The most important ingredients to that result are the ratio property for Delaunay complexes analyzed in detail by Talmor [5] and the weighted Delaunay complexes studied from an algorithmic viewpoint by Edelsbrunner and Shah [4]. Cheng et al. also give a linear time algorithm for constructing a weight assignment that satisfies Theorem 10.

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