Indicator Functions

`Topics:` inclusion-exclusion, truncation, inductive proof, unbounded convex polyhedra, restricting body, bounded convex polyhedra.

Inclusion-exclusion. Let H be a finite set of closed half-spaces in $\mathbb{R}^d$ and consider the convex polyhedron, $\bigcap H$, this set defines. The polyhedron may be bounded or unbounded as the one in Figure 1. For a subset

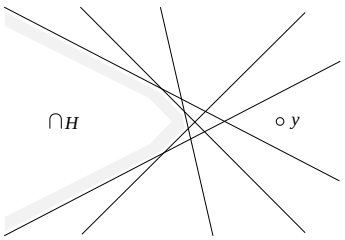

Figure 1: Five half-planes that intersect in an unbounded convex polygon. The point y lies outside all half-planes.

$X \subseteq H$ and a point $y \in \mathbb{R}^d$ define

$$\gamma_X(y) \quad = \quad \begin{cases} 1 & \text{if } x \notin h \text{ for all } h \in X, \\ 0 & \text{otherwise.} \end{cases}$$

We form an alternating sum of the γ that leads to an indicator function for the convex polyhedron. This is called the principle of inclusion-exclusion. Particularly, define

$$\Gamma_{2^H}(y) \quad = \quad \sum_{X \subseteq H} (-1)^{\operatorname{card} X} \cdot \gamma_X(y). \qquad (1)$$

The range of subsets X includes the empty set, for which $\gamma_\emptyset(y) = 1$ for all points y. We show that the non-zero terms in the sum cancel unless there is only one such contribution, which comes from the empty set. Indeed, define $G = \{h \in H \mid x \notin h\}$ and $m = \operatorname{card} G$. Then

$$\Gamma_{2^H}(y) = \Gamma_{2^G}(y) = \sum_{i=0}^{m} (-1)^i \binom{m}{i} = (1-1)^m = 0,$$

unless $G = \emptyset$, in which case we get $\Gamma_{2^H}(y) = 1$. In words, Γ_{2^H} is the indicator function for $\bigcap H$.

LEMMA 1. $\Gamma_{2^H}(y) = \begin{cases} 1 & \text{if } x \in \bigcap H, \\ 0 & \text{if } y \notin \bigcap H. \end{cases}$

Truncation. Most of the terms in the exponentially long formula (1) are redundant and can be removed. Specifically, we only keep terms that correspond to faces of $\bigcap H$. A *face* is the intersection of the polyhedron with a supporting hyperplane. We get each face as intersection of the polyhedron with a subset of the hyperplanes bounding half-spaces in H:

$$F_X \quad = \quad \bigcap H \cap \bigcap_{h \in X} \operatorname{bd} h.$$

For $X = \emptyset$ we get $F_\emptyset = \bigcap H$, which we consider an improper face but still a face of $\bigcap H$. It is convenient to assume general position, which in this context means that there are no two subsets $X \neq Y$ of H that define the same non-empty face $F_X = F_Y$. Let $D = D(H) = \{X \subseteq H \mid F_X \neq \emptyset\}$ be the system of subsets that define non-empty faces. For sets X in D there is an intuitive interpretation of $\gamma_X(y)$. Consider F_X *visible* from y if y sees all containing facets from outside the polyhedron. According to this definition the faces on the silhouette are not visible. Then $\gamma_X(y) = 1$ iff F_X is visible from y. The restriction of the inclusion-exclusion formula (1) to the system D is

$$\Gamma_D(y) \quad = \quad \sum_{X \in D} (-1)^{\operatorname{card} X} \cdot \gamma_X(y). \qquad (2)$$

We claim that Γ_D is still the indicator function of $\bigcap H$, even though the number of terms is dramatically smaller than that of Γ_{2^H}.

Inductive proof. This claim is sufficiently important to warrant the presentation of a complete proof.

THEOREM 2A. $\Gamma_D(y) = \begin{cases} 1 & \text{if } y \in \bigcap H, \\ 0 & \text{if } y \notin \bigcap H. \end{cases}$

PROOF. We use induction over the cardinality of the set $G = \{h \in H \mid y \notin h\}$. The basis of the induction is covered by the case where G is empty. In this case $y \in \bigcap H$ and $\Gamma_D(y) = \gamma_\emptyset(y) = 1$ as required. Assume $G \neq \emptyset$, let $g \in G$, and define $\bar{g}$ as the closed complement of g, which is another half-space, but one not in H. Define sets of half-spaces $H' = H - \{g\}$ and $H'' = H' \cup \{\bar{g}\}$. The corresponding systems are $D' = D(H')$ and $D'' = D(H'')$. The convex polyhedron $\bigcap H'$ is obtained by removing the constraint g, and therefore $\bigcap H' = \bigcap H \cup \bigcap H''$, see Figure 2. We

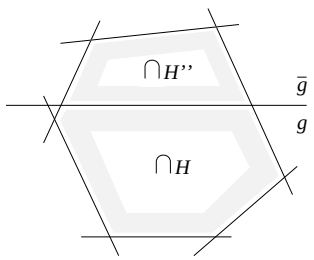

Figure 2: The union of $\bigcap H$ and $\bigcap H''$ is $\bigcap H'$.

distinguish three types of faces of $\bigcap H'$, the ones contained in g, the ones crossing the hyperplane shared by g and $\bar{g}$, and the ones contained in $\bar{g}$. The corresponding systems form a partition $D' = L' \dot\cup M' \dot\cup U'$, where

$$\begin{aligned} L' &= \{X \in D \mid F_X \subseteq g\}, \\ M' &= \{X \in D \mid F_X \cap g \neq \emptyset \text{ and } F_X \cap \bar{g} \neq \emptyset\}, \\ U' &= \{X \in D \mid F_X \subseteq \bar{g}\}. \end{aligned}$$

The faces of $\bigcap H$ are defined by sets in L', M' and M, and the faces of $\bigcap H''$ are defined by sets in U', M' and M'', where

$$\begin{aligned} M &= \{X \cup \{g\} \mid X \in M'\}, \\ M'' &= \{X \cup \{\bar{g}\} \mid X \in M'\}. \end{aligned}$$

The introduced systems partition D', D, D''. We can therefore write the Γ values of D', D, D'' as sums of the Γ values of the five systems:

$$\begin{aligned} \Gamma_{D'} &= \Gamma_{L'} + \Gamma_{M'} + \Gamma_{U'}, \\ \Gamma_D &= \Gamma_{L'} + \Gamma_{M'} + \Gamma_M, \\ \Gamma_{D''} &= \Gamma_{M'} + \Gamma_{M''} + \Gamma_{U'}, \end{aligned}$$

and hence,

$$\Gamma_D = (\Gamma_{D'} - \Gamma_{D''}) + \Gamma_{M''} + (\Gamma_{M'} + \Gamma_M).$$

We argue that all three terms on the right side vanish. Both H' and H'' have one less half-space not containing y than H does. The induction hypothesis thus applies. By assumption $y \in \bar{g}$ which implies that $y \in \bigcap H'$ iff $y \in \bigcap H''$ and therefore $\Gamma_{D'}(y) - \Gamma_{D''}(y) = 0$. The second term vanishes because all sets in M'' contain $\bar{g}$. The third term vanishes because $X \in M'$ iff $X \cup \{g\} \in M$. We have $\gamma_X(y) = \gamma_{X \cup \{g\}}(y)$ and $\Gamma_{M'}(y) + \Gamma_M(y) = 0$ because the γ values cancel pairwise. □

Unbounded convex polyhedra. For an unbounded convex polyhedron we can find y so it lies outside all half-spaces in H, as in Figure 1. We fix y and rewrite the inclusion-exclusion formula in Theorem 2A in terms of face numbers. Let f_i be the number of i-dimensional faces of $\bigcap H$. The *Euler characteristic* of $\bigcap H$ is

$$\chi_{\bigcap H} = \sum_{i=0}^{d} (-1)^{d-i} \cdot f_i.$$

By assumption of general position, f_i is the number of sets $X \in D$ with cardinality $d - i$. We have $\gamma_X(y) = 1$ for all $X \in D$ and therefore $\Gamma_D(y) = (-1)^d \cdot \chi_{\bigcap H}$. This implies the Euler formula for unbounded convex polyhedra.

COROLLARY 3A. If $\bigcap H$ is unbounded then $\chi_{\bigcap H} = 0$.

We need a slightly stronger version of Theorem 2A to prove the Euler formula for bounded convex polyhedra.

Restricting body. Theorem 2A is weakened by restricting the points y inside a convex body A. This weaker result is then strengthened by eliminating terms that become redundant because of the restriction. Define

$$E = E(A, H) = \{X \in D \mid F_X \cap \operatorname{int} A \neq \emptyset\}.$$

We show that for points $y \in A$, Γ_E is the indicator function for $\bigcap H$.

THEOREM 2B. $\Gamma_E(y) = \begin{cases} 1 & \text{if } y \in A \cap \bigcap H, \\ 0 & \text{if } y \in A - \bigcap H. \end{cases}$

PROOF. We construct a convex polyhedron $\bigcap H_A$ that contains A and approximates A sufficiently closely. Specifically, we require $E(\bigcap H_A, H) = E(A, H)$, as in Figure 3. Define $D_A = D(H \cup H_A)$ and use Theorem 2A to get

$$\Gamma_{D_A}(y) = \begin{cases} 1 & \text{if } y \in \bigcap H_A \cap \bigcap H, \\ 0 & \text{if } y \notin \bigcap H_A \cap \bigcap H. \end{cases}$$

By choice of H_A, every point $y \in A$ is contained in all half-spaces of H_A. Hence $\gamma_X(y) = 0$ if $X \cap H_A \neq \emptyset$.

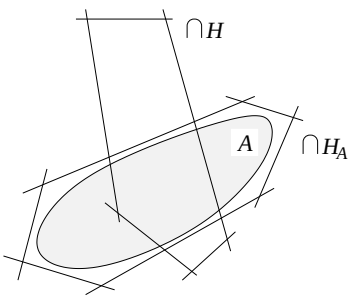

Figure 3: A face F of $\bigcap H$ intersects the interior of A iff it intersects the interior of $\bigcap H_A$.

The system E contains exactly all sets $X \in D_A$ for which $X \cap H_A = \emptyset$. Hence $\Gamma_E(y) = \Gamma_{D_A}(y)$ for all $y \in \bigcap H_A$ and therefore also for all $y \in A$. □

Bounded convex polyhedra. We return to the computation of the Euler characteristic, this time for a bounded convex polyhedron $\bigcap H$. We choose a line not parallel to any face of $\bigcap H$ and points y and z on the line sufficiently far out in the two directions. As illustrated in Figure 4, this partitions H into the set Y of half-spaces that do not contain y and the set Z of half-spaces that do not contain z. The proper faces of

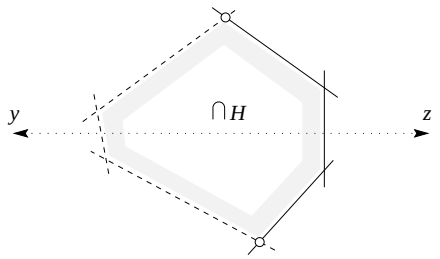

Figure 4: The boundary of $\bigcap Y$ is dashed and that of $\bigcap Z$ is solid. The two hollow vertices form the boundary of the silhouette.

$\bigcap H$ either belong to $\bigcap Y$ or to $\bigcap Z$ or to the silhouette as seen in a view parallel to the line yz. Let y_i be the number of i-dimensional faces of $\bigcap Y$ that have non-empty intersection with the interior of $\bigcap Z$, and define z_i symmetrically. Let s_i be the number of i-dimensional faces of the silhouette. The projection of the silhouette onto a hyperplane normal to yz is a bounded convex polyhedron $\bigcap S$ of dimension $d-1$.

COROLLARY 3B. If $\bigcap H$ is bounded and $\dim \bigcap H = d$ then $\chi_{\bigcap H} = (-1)^d$.

PROOF. We use induction. For $d = 1$ the polyhedron is a closed interval with $\chi = 1-2 = -1$, which establishes the induction basis. For $d > 1$ we have

$$\begin{aligned} \chi_{\bigcap H} &= \sum_{i=0}^{d} (-1)^{d-i} \cdot f_i \\ &= \sum_{i=0}^{d} (-1)^{d-i} \cdot (y_i + z_i + s_i). \end{aligned}$$

Observe that this sum counts the d-face $\bigcap H$ the correct number of times, namely $y_d + z_d - s_{d-1} = 1$ time. We get $\sum (-1)^{d-i} y_i = \sum (-1)^{d-i} z_i = 0$ by Theorem 2B and $\sum (-1)^{d-1-i} s_i = (-1)^{d-1}$ by induction hypothesis. This implies $\chi_{\bigcap H} = -(-1)^{d-1}$ as required. □

The fact that the Euler characteristic of a bounded convex polyhedron is $(-1)^d$ should not be surprising. The alternating sum of Betti numbers of a convex polyhedron is always $\beta_0 = 1$. Because the definition of the Euler characteristic given above is based on codimensions, and that of $\sum \beta_i$ is based on dimensions, we get the opposite sign in every odd dimension.

Bibliographic notes. Most of the material in this section is taken from Edelsbrunner [2], who uses inclusion-exclusion to measure a union of spherical balls. As demonstrated, this principle also yields the Euler formula for convex polyhedra. There are many proofs of that formula, and the one in these notes is one of the few (the only one?) that uses induction in an essential manner. The first proof of Euler's formula for convex polyhedra in general dimension d goes back to Ludwig Schläfli [3]. He implicitly assumes that boundary complexes of convex polyhedra are shellable. Bruggesser and Mani [1] prove in 1972 that this is really so and thus fill the gap left by Schläfli.

[1] H. Bruggesser and P. Mani. Shellable decompositions of cells and spheres. *Math. Scand.* **29** (1972), 197–205.

[2] H. EDELSBRUNNER. The union of balls and its dual shape. *László Fejes Tóth Festschrift*, eds. I. Bárány and J. Pach, *Discrete Comput. Geom.* **13** (1995), 415–440.

[3] L. SCHLÄFLI. Theorie der vielfachen Kontinuität. Written 1850–52 and published in *Denkschriften der Schweizerischen naturforschenden Gesellschaft* **38** (1901), 1–237.