Curvature

`Topics:` plane curves, curvature, space curves, torsion, surfaces, principal curvature, mean and Gaussian curvature.

Plane curves. A *map* is a continuous function. A *curve* in the plane is a map $\gamma : \mathbb{R} \to \mathbb{R}^2$. Depending on the context, we can use other 1-dimensional domains in the definition of a curve. For example, $\gamma : \mathbb{S}^1 \to \mathbb{R}^2$ is a closed curve and $\gamma : [0,1] \to \mathbb{R}^2$ is a curve with boundary. The curve is *smooth* if derivatives of all orders exist. Usually we need only a small number of derivatives, but out of laziness it is convenient to first assume they all exist. The first derivative exaggerates the infinitesimal difference between two vectors defined for arguments that are infinitesimally apart. The result is a vector $\dot{\gamma} = \frac{\mathrm{d}\gamma}{\mathrm{d}x}$ tangent to the curve at x, see Figure 1. The direction of the vector specifies the best line approximating the curve in the neighborhood of x. The length of the vector is the speed of the parametrization. It is convenient to assume unit speed everywhere. Under this assumption, the first derivative is the unit *tangent vector* denoted as $T = \dot{\gamma}$.

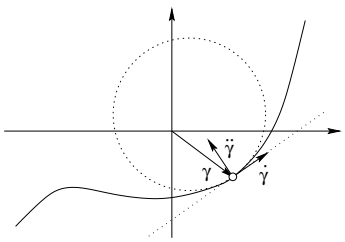

Figure 1: The dotted line and circle are the best approximations of their kind at the point where they touch the curve.

Curvature. The second derivative is related to the *osculating circle*, which is the best circle approximating the curve at a point. Define $\ddot{\gamma} = \frac{\mathrm{d}T}{\mathrm{d}x}$. We visualize this construction by anchoring the tangent vector at the origin. By assumption of unit speed, the tip of the tangent vector moves only on the unit circle. Hence, T is a map $\mathbb{R} \to \mathbb{S}^1$, which is often referred to as the *Gauss map* of the curve, see Figure 2. The second derivative is tangent to the circle and therefore necessarily normal to T. The length of $\ddot{\gamma}(x)$ is one over the radius of the osculating circle at x. The *curvature* at x is one over the radius of that circle: $\kappa(x) = \|\ddot{\gamma}(x)\|$. The unit *normal vector* is the normalized second derivative and is denoted as $N = \frac{\ddot{\gamma}}{\kappa}$. In the plane we can distinguish a left from a right turn and define a signed curvature accordingly. Points with zero curvature are either inflection points, where the curvature changes sign, or points with tangent lines of high contact.

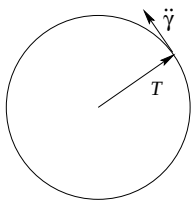

Figure 2: The tangent vector defines the Gauss map whose image lies on the unit circle.

Space curves. A *space curve* is a map $\gamma : \mathbb{R} \to \mathbb{R}^3$. Assuming unit speed we can still define $T = \dot{\gamma}$ and $N = \frac{\dot{T}}{\kappa}$. The Gauss map $T : \mathbb{R} \to \mathbb{S}^2$ is now a map to the unit sphere, and N is a tangent to that sphere, see

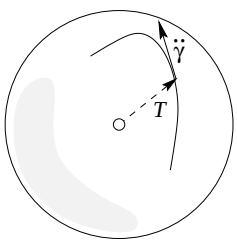

Figure 3: The normal to the curve is the tangent to the Gauss map.

Figure 3. If we reparametrize the curve so the Gauss map is unit speed then we get the third derivative as a vector normal to both T and N. The normalized version of that third derivative is the unit *binormal* $B = T \times N$. The three unit vectors define a local coordinate system known as the *Frenet frame*, see Figure 4.

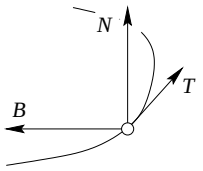

Figure 4: Frenet frame at a point of a space curve.

Torsion. The *normal plane* at $\gamma(x)$ is spanned by N and B. The *osculating plane* is spanned by T and N; it is the plane of locally highest contact. From the length of the derivative of the normal vector we get another interesting number. Since $\dot{N}$ is normal to N we have $\dot{N} = \lambda \cdot T + \tau \cdot B$ for suitable coefficients λ and τ. But $T \cdot N = 0$ and therefore

$$\begin{aligned} 0 &= \dot{T} \cdot N + T \cdot \dot{N} \\ &= \kappa \cdot N \cdot N + T \cdot (\lambda \cdot T + \tau \cdot B) \\ &= \kappa + \lambda. \end{aligned}$$

Hence,

$$\dot{N} = -\kappa \cdot T + \tau \cdot B$$

for some real $\tau = \tau(x)$ called the *torsion* of the curve at x. It measures the rolling motion we experience as we move along γ. As long as $\kappa \neq 0$, τ is a smooth map from $\mathbb{R}$ to $\mathbb{R}$.

Surfaces. A *surface* is a map from a 2-dimensional domain to $\mathbb{R}^3$. For specificity we assume the domain is an open disk in $\mathbb{R}^2$, and because open disks are homeomorphic to the entire plane we simply write $f : \mathbb{R}^2 \to \mathbb{R}^3$. The surface is *smooth* if derivatives of all orders exist. Derivatives are computed along curves in the surface and combined as needed. For example, the tangent plane at a point is spanned by the tangent vectors of two curves passing through that point. This is illustrated in Figure 5, where f^{-1} is a simple projection to the plane.

Similarly, curvature on the surface at a point is computed from sectional curvatures. At a point $x \in \mathbb{R}^2$ we consider all curves passing through x, and for each curve γ we consider the space curve $f(\gamma) = f \circ \gamma$ passing through $f(x)$. Such a space curve is locally a *geodesic* if its normal vector at x agrees with the surface normal at x. All space curves that are locally geodesic and pass with unit speed through x in the direction of a fixed tangent vector v have the same curvature at x, referred to as the *normal curvature* and denoted as $\kappa_v(x)$. This is the amount of curvature forced upon the space curve by the bending of the surface.

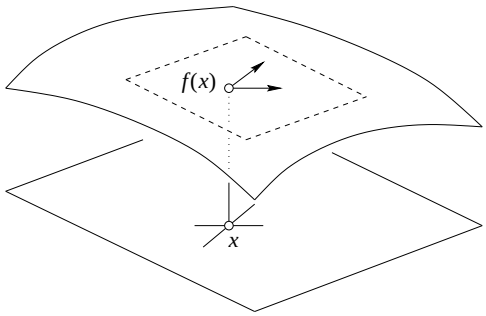

Figure 5: The tangent vectors spanning the tangent plane at x.

Principal curvature. The tangent vectors v form a unit circle in the tangent plane that passes through $f(x)$. The *principal curvatures* are the maximum and minimum normal curvatures at x, denoted as $\kappa_1(x) \geq \kappa_2(x)$. Let v_1 and v_2 be the vectors defining the two principal curvatures. Euler proved hundreds of years ago that v_1 and v_2 are orthogonal and that they determine the curvature values along all other tangent vectors at x.

THEOREM 1. Let $v = \sin\alpha \cdot v_1 + \cos\alpha \cdot v_2$. Then the normal curvature along v is

$$\kappa_v = \sin^2\alpha \cdot \kappa_1 + \cos^2\alpha \cdot \kappa_2.$$

The vectors defining principal curvatures are tangent to the *principal curvature lines*, which are space curves in the surface. An *umbilic* point is one where the two principal curvatures are the same: $\kappa_1 = \kappa_2$. At such points the directions of the principal curvature lines are ambiguous. For example, the sphere has only umbilic points and therefore no principal curvature lines.

Mean and Gaussian curvature. Other popular expressions of curvature on a surface are the *mean* and the *Gaussian curvature*,

$$\begin{aligned} H &= \frac{\kappa_1 + \kappa_2}{2}, \\ K &= \kappa_1 \cdot \kappa_2. \end{aligned}$$

Mean curvature plays a fundamental role in the determination of minimal surfaces: the mean curvature be everywhere 0. The Gaussian curvature is the only one that can be defined intrinsically, which means it does not depend on the embedding of the surface. To describe this important property in technical terms we define the *distance* between two points on the surface as the length of the shortest path connecting the points. An *isometry* from one surface to another is a map that preserves distances. For example, there is an isometry between a piece of a cylinder and a piece of the plane. The celebrated theorema egregium of Friedrich Gauß is

THEOREM 2. Gaussian curvature is an isometric invariant.

In other words, Gaussian curvature can be defined without reference to the embedding of the surface. All we need to be able to do is measure length and do mathematics. Specifically, construct the curve whose points are at distance $\varepsilon > 0$ from a point p in the surface, and denote its length by $L_p(\varepsilon)$. If the surface is a plane then $L_p(\varepsilon) = 2\pi\varepsilon$. As it turns out, the Gaussian curvature is $-\frac{1}{3}$ times the second derivative of the ratio $L_p(\varepsilon)$ over $2\pi\varepsilon$.

Bibliographic notes. Curvature of smooth curves and surfaces is a classic topic in differential geometry and discussed in all major textbooks in that area. We recommend the text by Barrett O'Neill [2] for a reasonably comprehensive introduction to that subject, and the book by Bruce and Giblin [1] for an accessible description of the connection between classic differential geometry and the more advanced singularity theory. The intrinsic computation of the Gaussian curvature as the second derivative of a ratio of lengths is taken from Thurston [3, p. 47].

[1] J. W. BRUCE AND P. J. GIBLIN. *Curves and Singularities.* Second edition, Cambridge Univ. Press, England, 1992.

[2] B. O'NEILL. *Elementary Differential Geometry.* Second edition, Academic Press, San Diego, California, 1997.

[3] W. P. THURSTON. *Three-dimensional Geometry and Topology. Vol. 1.* Edited by Silvio Levy, Princeton Univ. Press, New Jersey, 1997.