Mixed Complex

`Topics:` shrinking cells, skin decomposition, two circles, three circles.

Shrinking cells. Given a finite set of circles S, the (weighted) Voronoi regions cover the entire plane. The (weighted) Delaunay triangulation is the dual and its triangles cover the convex hull of the circle centers. We shrink each Voronoi region by a factor s towards the center of its generating circle. Symmetrically, we shrink every Delaunay triangle by a factor $t = 1-s$ towards the dual Voronoi vertex. At the end of the shrinking process we have a collection of non-overlapping convex polygonal cells. By construction, the shrunken Delaunay triangles touch the shrunken Voronoi regions at common corner points, as illustrated in Figure 13. The gaps between the shrunken Voronoi regions and Delaunay triangles are filled by rectangular cells. We can think of each rectangle as the Cartesian product of a shrunken Delaunay edges times the shrunken dual Voronoi edge. The three types of cells make up the *mixed complex*.

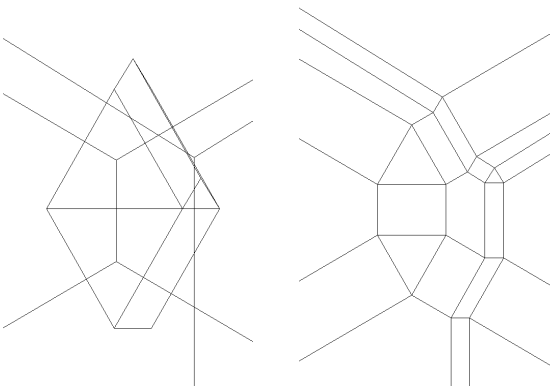

Figure 13: Voronoi diagram and Delaunay triangulation to the left and mixed complex to the right.

Skin decomposition. The reason for introducing the mixed complex is that it provides an effective algorithm to computing the skin of S. We first explain how this works and later discuss why it works. We have three cases, one for each type of cell in the mixed complex. The labeling of the cases alludes to the generalization of coaxal systems discussed in Section 25. All three cases occur in the example shown in Figure 14.

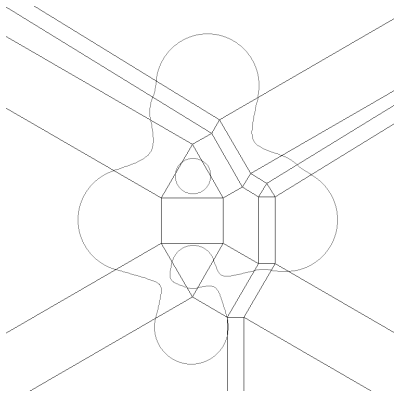

Figure 14: Skin for $s = \frac{1}{2}$ and mixed complex decomposing it into circular and hyperbolic arcs.

`Case 1/3.` The mixed cell is a Voronoi region $V_{\hat{p}}$ shrunken by a factor s towards p:

$$C \quad = \quad s \cdot V_{\hat{p}} + (1-s) \cdot p.$$

Within C the skin is the same as the shrunken circle: $C \cap \mathrm{skn}^s S = C \cap \hat{p}^s$.

`Case 2/2.` The mixed cell C is a Voronoi edge, $V_{\hat{p}} \cap V_{\hat{q}}$, shrunken by a factor s, times the dual Delaunay edge, pq, shrunken by a factor $t = 1 - s$. Within C the skin is the common envelope of F^s and G^t, where $F = \mathrm{aff}\,\{\hat{p}, \hat{q}\}$ and G is the line of orthogonal circles.

`Case 3/1.` The mixed cell is a Delaunay triangle pqr shrunken by a factor t towards the dual Voronoi vertex $x = V_{\hat{p}} \cap V_{\hat{q}} \cap V_{\hat{r}}$:

$$C \quad = \quad t \cdot pqr + (1-t) \cdot x.$$

By construction, x has the same weighted distance X^2 from all three circles. In other words, $\hat{x} = (x, X^2)$ is orthogonal to $\hat{p}, \hat{q}, \hat{r}$. Within C the skin is the same as the shrunken orthogonal circle: $C \cap \mathrm{skn}^s S = C \cap \hat{x}^t$.

Two circles. Instead of arguing the correctness of the case analysis in general, we revisit the case of two circles, and after that we discuss the case of three circles. Think of the Delaunay edge pq as part of the affine hull of the two corresponding circles, $F = \mathrm{aff}\,\{\hat{p}, \hat{q}\}$. Assume that $\hat{p}$ and $\hat{q}$ intersect in two points and the envelope of the shrunken convex hull is a dumbbell, like the one in Figure 15. Where exactly do the circular arcs connect with the hyperbola arcs forming the neck? To answer this question we consider the dual Voronoi edge, $V_{\hat{p}} \cap V_{\hat{q}}$. The edges is really a line and we think of it as the line G of circles orthogonal to the circles in F. The circular arc contributed by $\hat{p}$ ends at the point a where $\hat{p}$ touches the envelope of F^s. Let $\hat{x}^t$ be the shrunken circle in G^t that touches the hyperbola at the same point a, but from the other side, as shown in Figure 15. By Lemma 6, the radius of $\hat{x}$ is $X = \sqrt{t/s} \cdot P$.

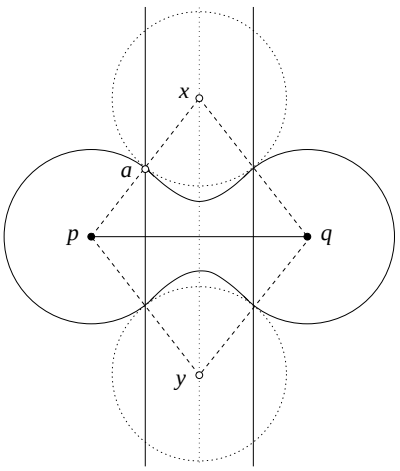

Figure 15: The infinite strip is the shrunken Delaunay edge times the (shrunken) infinite Voronoi edge.

The distances between p and a and between x and a are $\|p - a\| = \sqrt{s} \cdot P$ and $\|x - a\| = \sqrt{t} \cdot X = t/\sqrt{s} \cdot P$. The ratio is $\|p - a\|/\|x - a\| = s/t$. It follows that the line parallel to and at a ratio s/t between p and the Voronoi edge cuts the dumbbell where it changes from circular to hyperbola arcs. By symmetry we get another line between q and the Voronoi edge. The strip between the two lines is the mixed cell of Case 2/2.

Three circles. The interesting case is where the three disks bounded by $\hat{p}, \hat{q}, \hat{r}$ overlap pairwise but their common intersection is empty. This is the case illustrated in Figure 16. Let x be the dual Voronoi vertex and $\hat{x}$

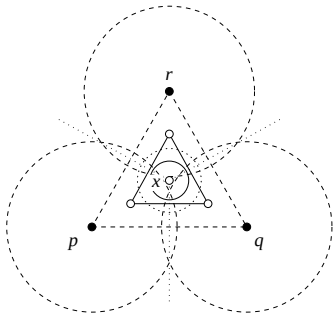

Figure 16: The dashed and dotted circles are orthogonal. The solid circle is part of the skin.

the circle orthogonal to $\hat{p}, \hat{q}, \hat{r}$. If $\hat{x}^t$ lies completely inside the shrunken triangle pqr then the convex hull of $\{\hat{p}, \hat{q}, \hat{r}\}$ contains circles that are orthogonal to $\hat{x}$ with radius $\sqrt{t/s}$ times the radius of $\hat{x}$ in every direction around x. After shrinking $\hat{x}$ to $\hat{x}^t$ and the circles in F to F^s we have a circle touching $\hat{x}^t$ in every direction. Hence, $\hat{x}^t$ is part of the s-skin of $\{\hat{p}, \hat{q}, \hat{r}\}$. The circle smoothes out the sharp angles at the corners of the triangular hole. In the case where $\hat{x}^t$ is partially outside the shrunken triangle, the smoothed version of the triangular hole consists of circular arcs inside and hyperbola arcs outside the shrunken triangle.

Bibliographic notes. Everything described in this and the preceding sections extends to three and higher dimensions. The theory of skin surfaces is fully described in Edelsbrunner [1]. The wealth of concrete information available for skin surfaces suggests they be used in mesh generation problems for domains with smooth but possibly complicated boundaries. The skin surface can be maintained under motion, which makes it a candidate for simulation problems where the domain changes with time.

[1] H. Edelsbrunner. Deformable smooth surface design. *Discrete Comput. Geom.* **21** (1999), 87–115.