Anisotropic Voronoi Diagrams and Guaranteed−Quality Anisotropic Mesh Generation

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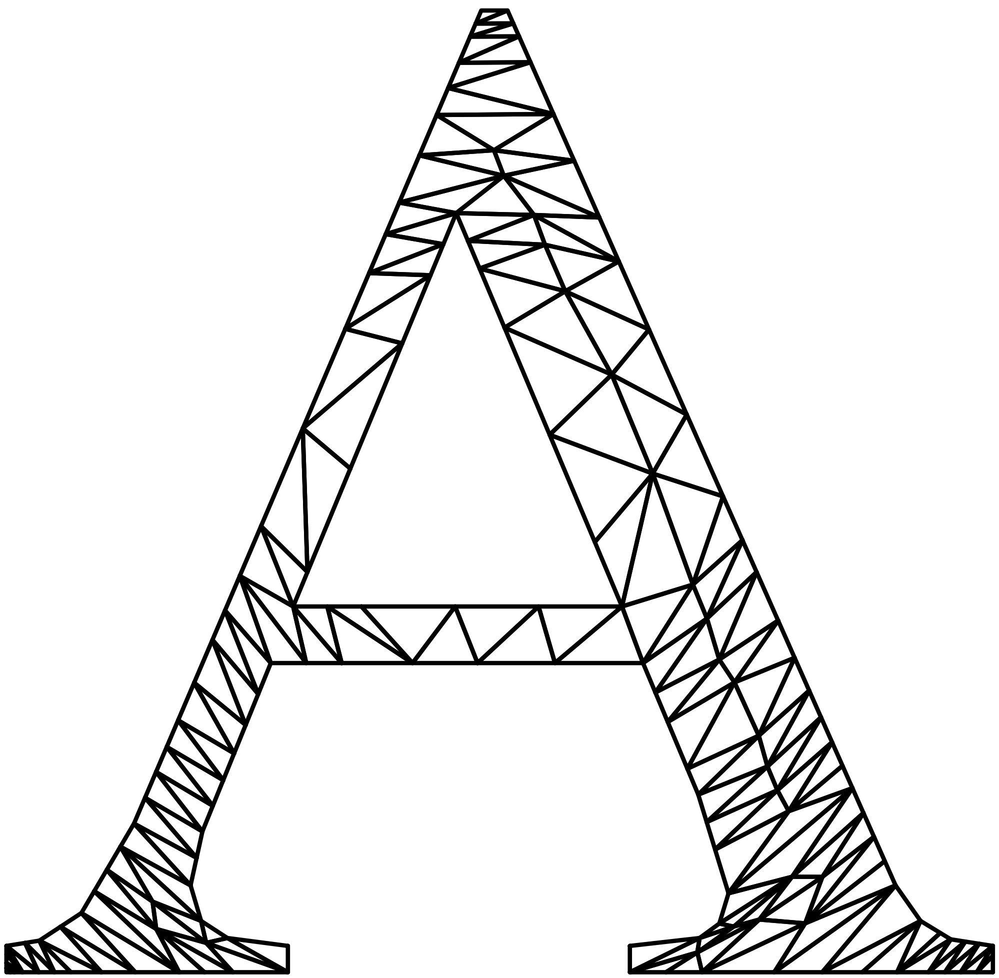
I. Anisotropic Meshes

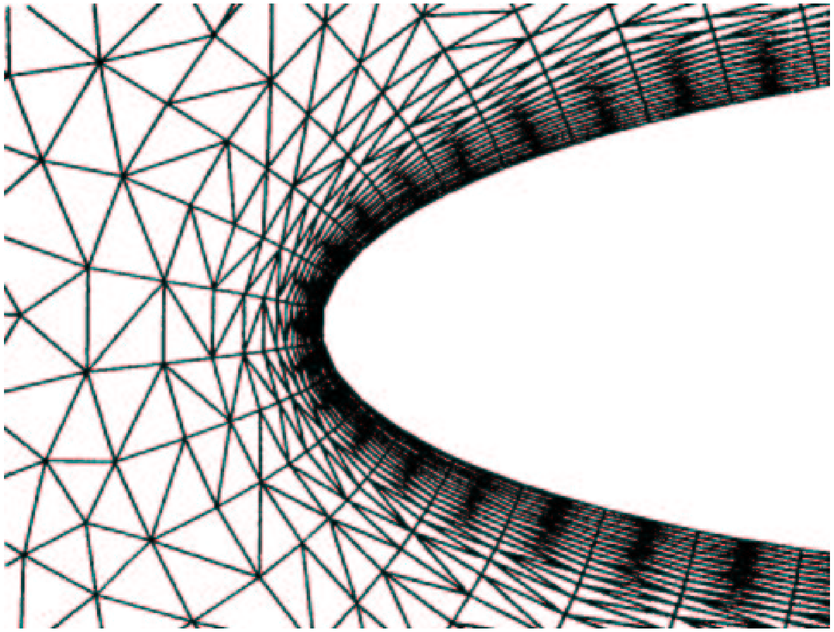


What Are Anisotropic Meshes?

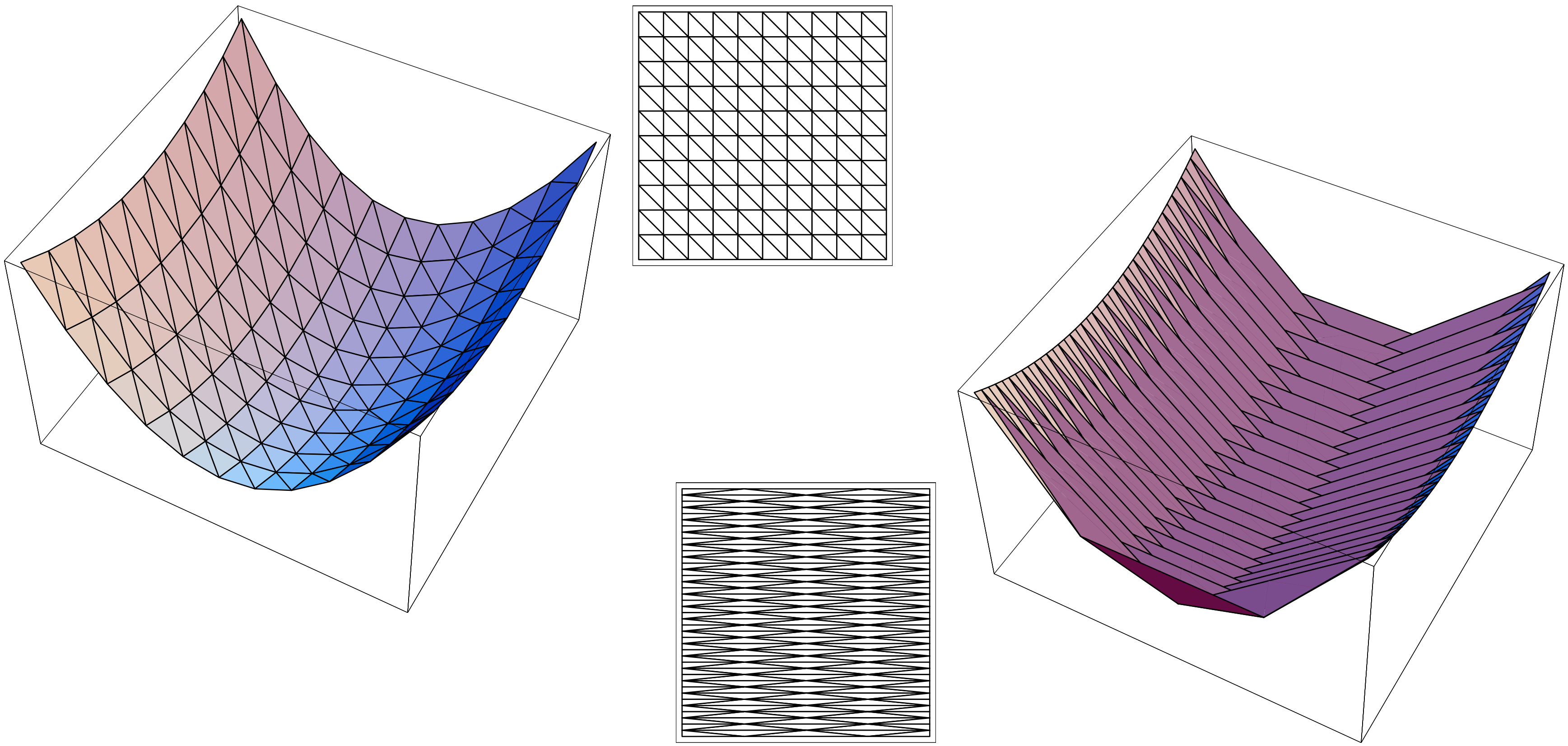
Meshes with long, skinny triangles (in the right places).

Why Are They Important?

- Often provide better interpolation of multivariate functions with fewer triangles.
- Used in finite element methods to resolve boundary layers and shocks.



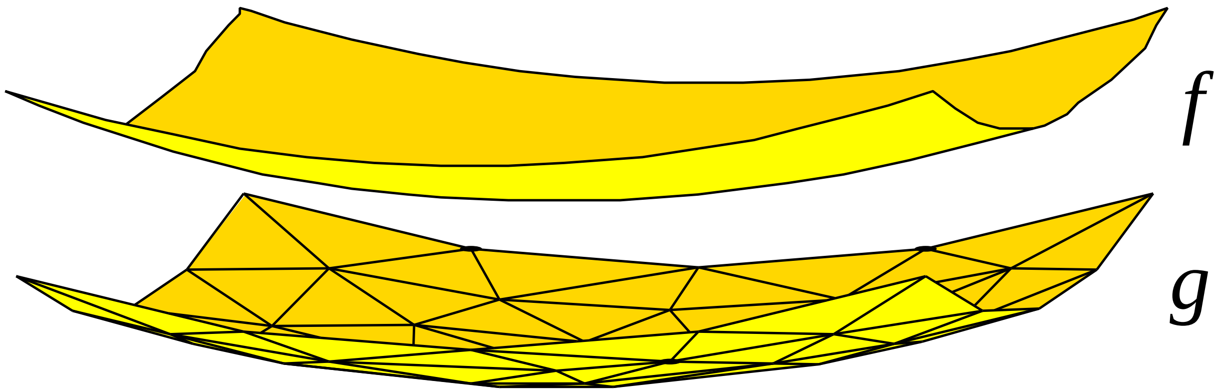
Source: ‘‘Grid Generation by the Delaunay Triangulation,’’ Nigel P. Weatherill, 1994.



Triangle shape is critical for

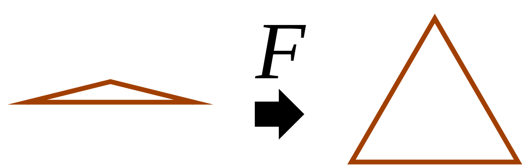
- surface triangulations in graphics;
- finite element meshes in physical modeling;
- triangulations in interpolation.

Interpolation of Functions with Anisotropic Curvature



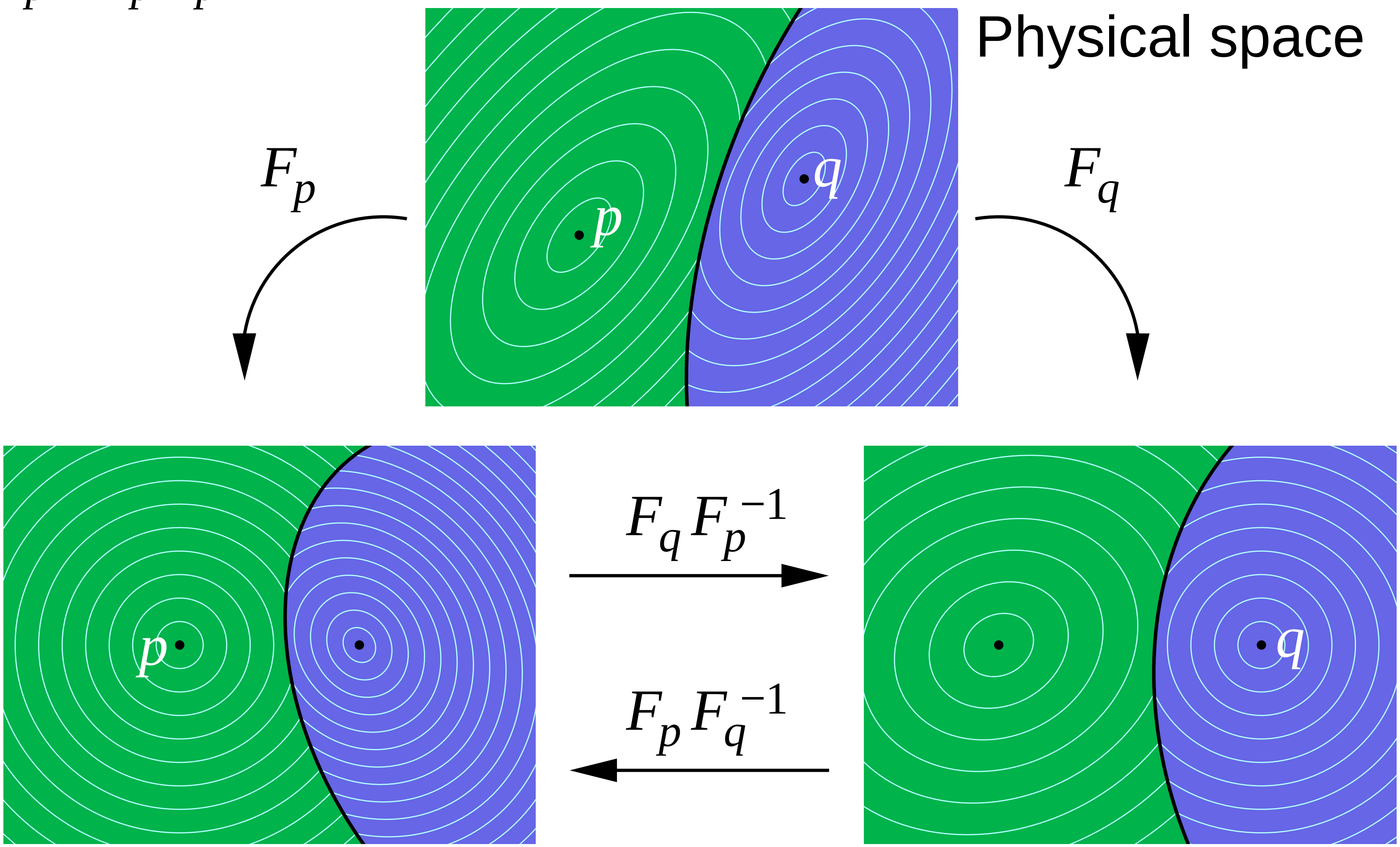
H = Hessian of f. Let $F^2 = H$ with F symmetric pos−def.

You can judge the quality of a triangle t by checking if Ft is "round."



Distance Measures

Metric tensor M_p: distances & angles measured by p.
Deformation tensor F_p: maps physical to rectified space.
$M_p = F_p^T F_p$.



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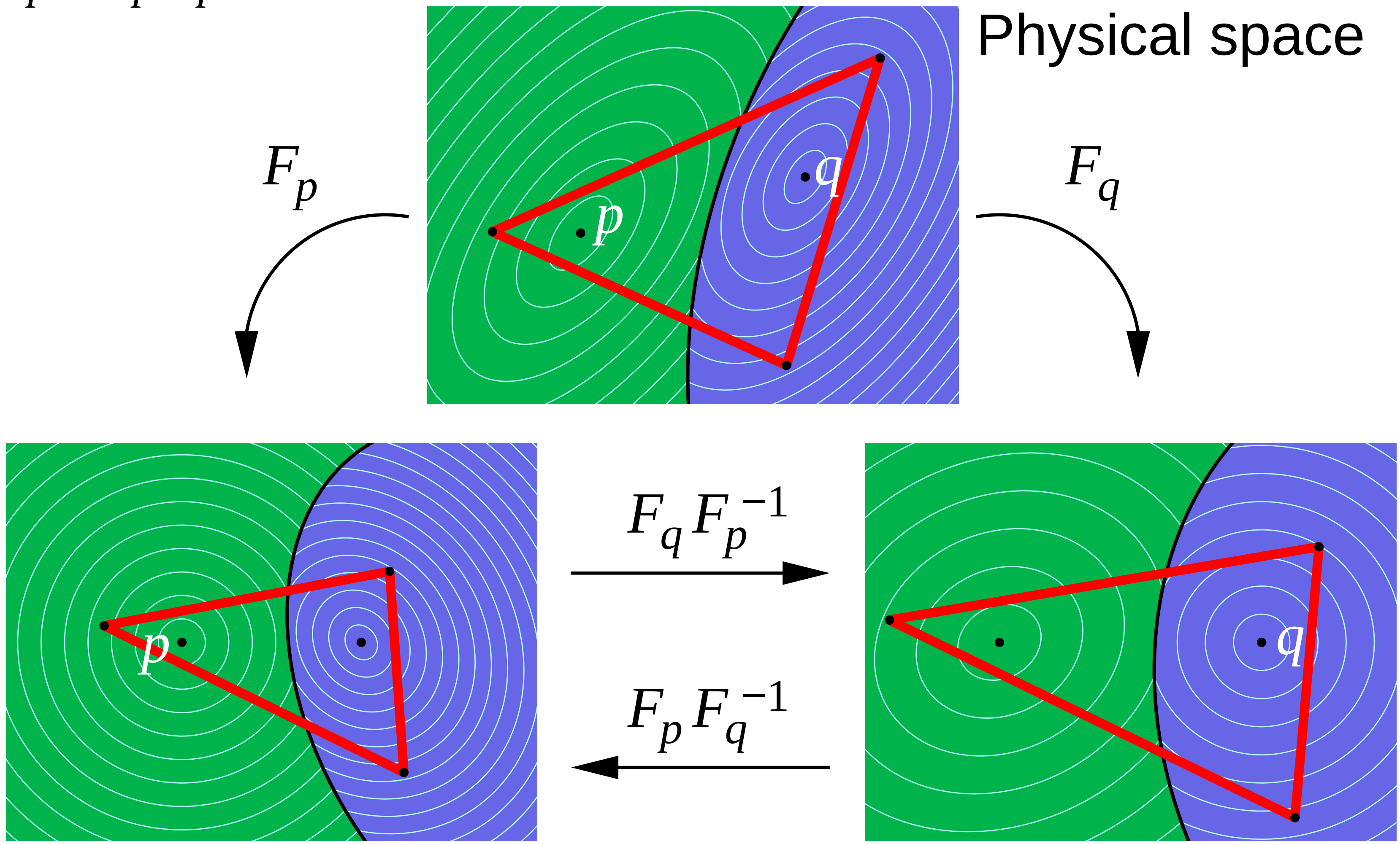
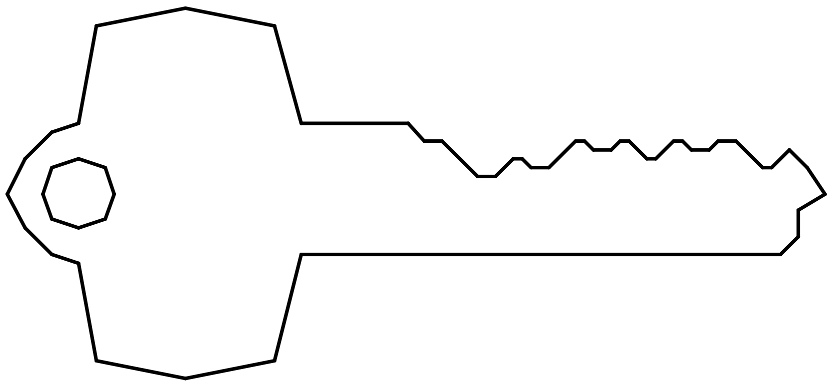
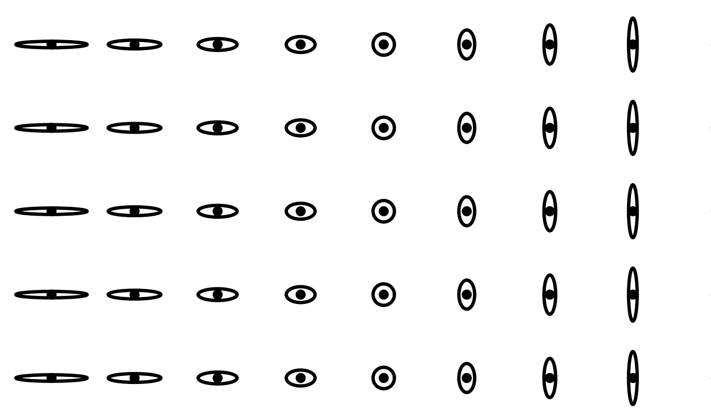


Every point wants to be in a “nice” triangle in rectified space.

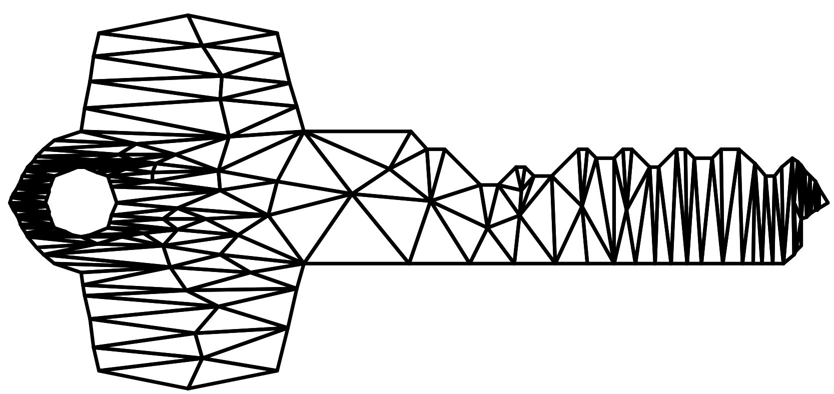
The Anisotropic Mesh Generation Problem

Given polygonal domain and metric tensor field *M*,





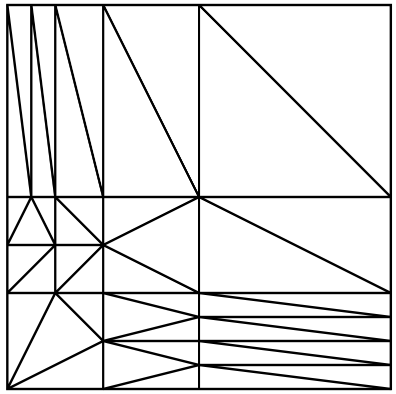
generate anisotropic mesh.



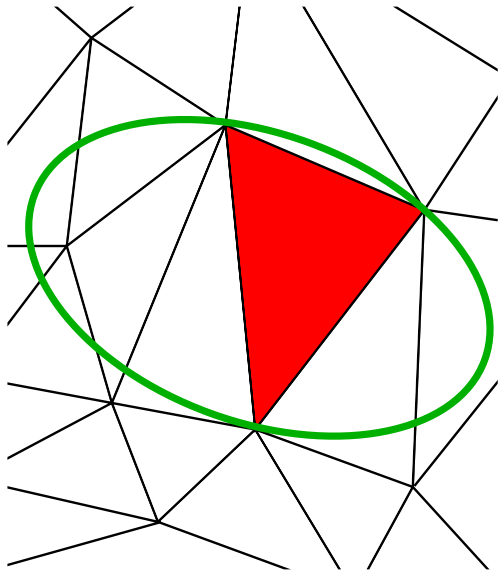
A Hard Problem (Especially in Theory)

Common approaches to guaranteed−quality mesh generation do not adapt well to anisotropy.

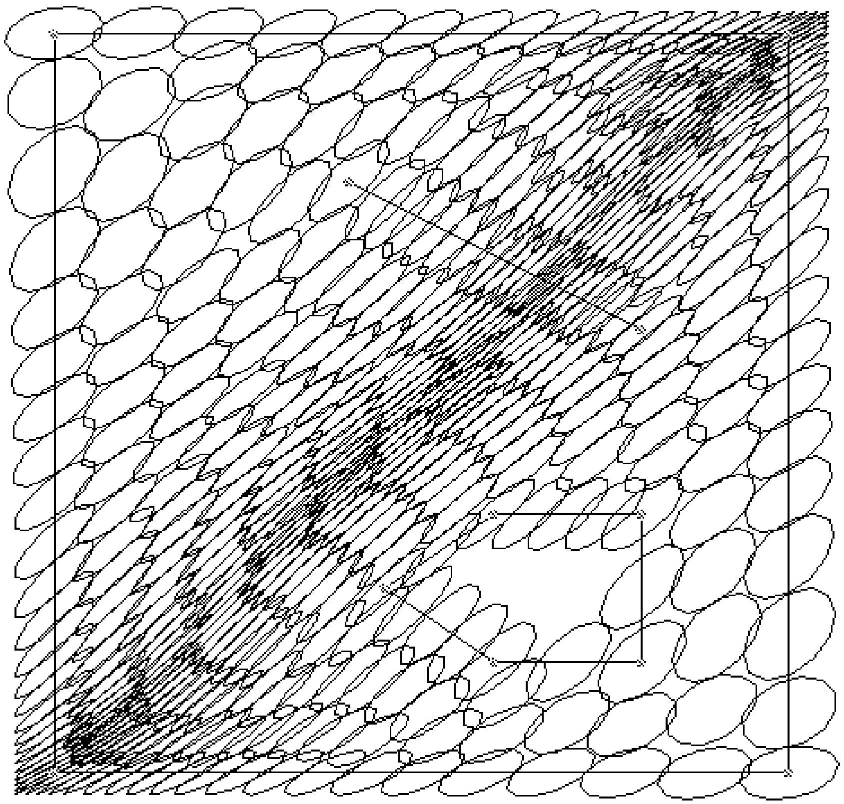
- Quadtree−based methods can be adapted to horizontal and vertical stretching, but not to diagonal stretching.

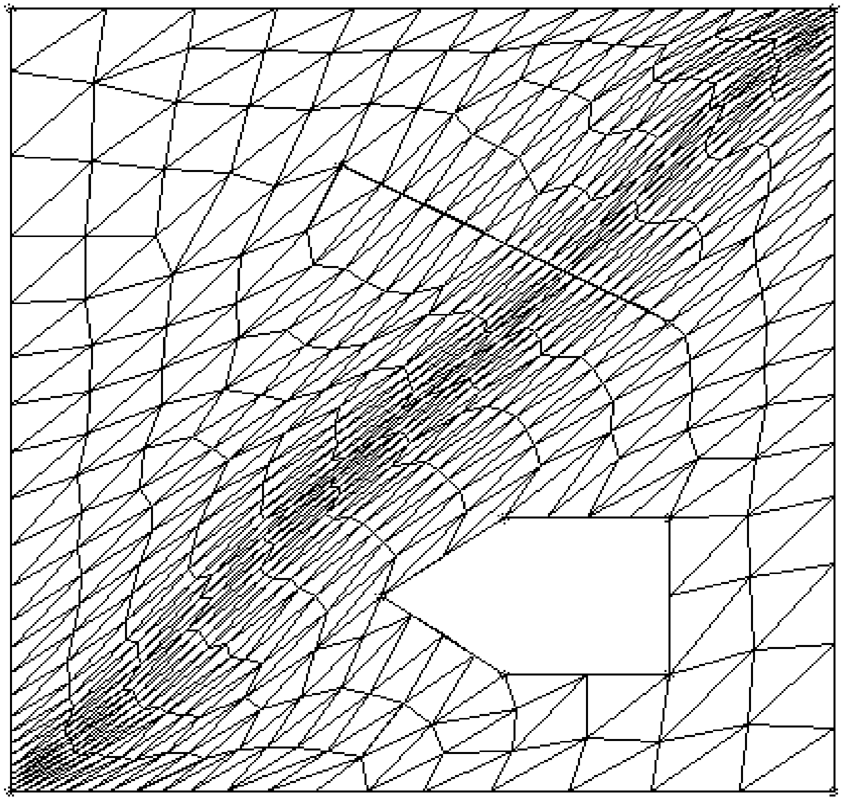


- Delaunay triangulations lose their global optimality properties when adapted to anisotropy. No ‘‘empty circumellipse’’ property.



Heuristic Algorithms for Generating Anisotropic Meshes





Bossen−Heckbert [1996]

Shimada−Yamada−Itoh [1997]

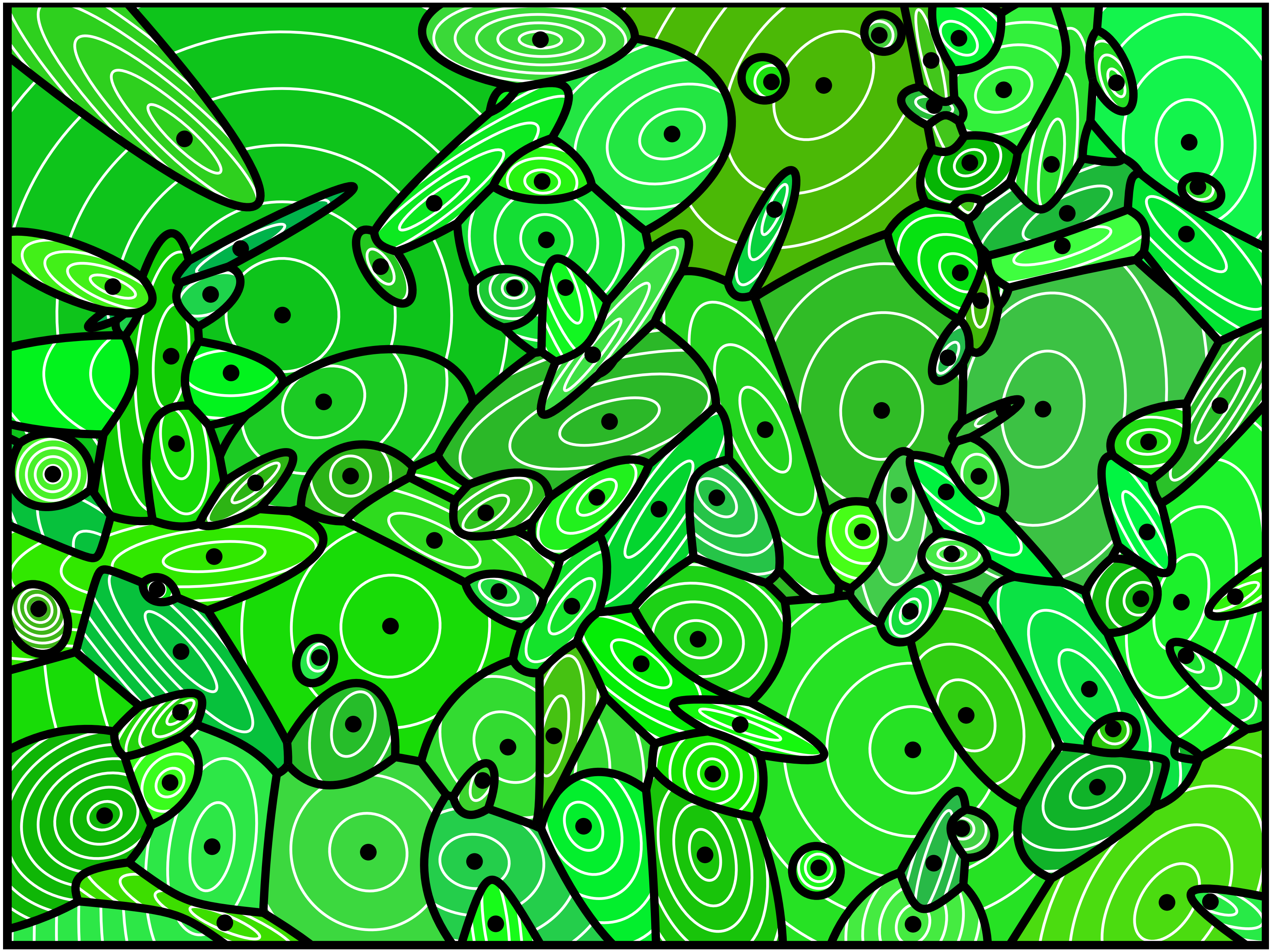
George−Borouchaki [1998]

Li−Teng−Üngör [1999]

Our Solution

- We tried to invent an “‘anisotropic Delaunay triangulation’’ that is always well defined. We couldn’t do it. So...
- Our meshing algorithm refines a special, anisotropic kind of Voronoi diagram.
- No triangulation until the very end.

II. Anisotropic Voronoi Diagrams



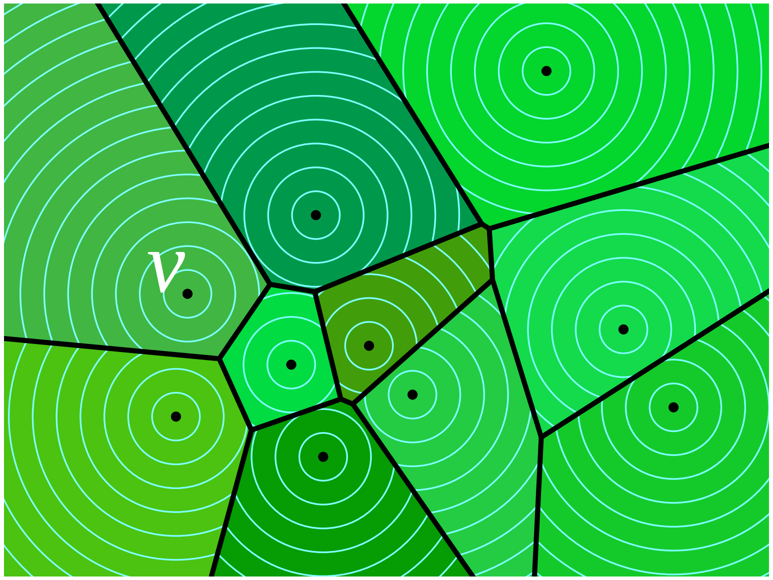
Voronoi Diagram: Definition

Given a set V of sites in E^d, decompose E^d into cells. The cell Vor(v) is the set of points "closer" to v than to any other site in V.

Mathematically:

Vor(v) = $\{p$ in E^d: $d_v(p) \leq d_w(p)$ for every w in V.$\}$

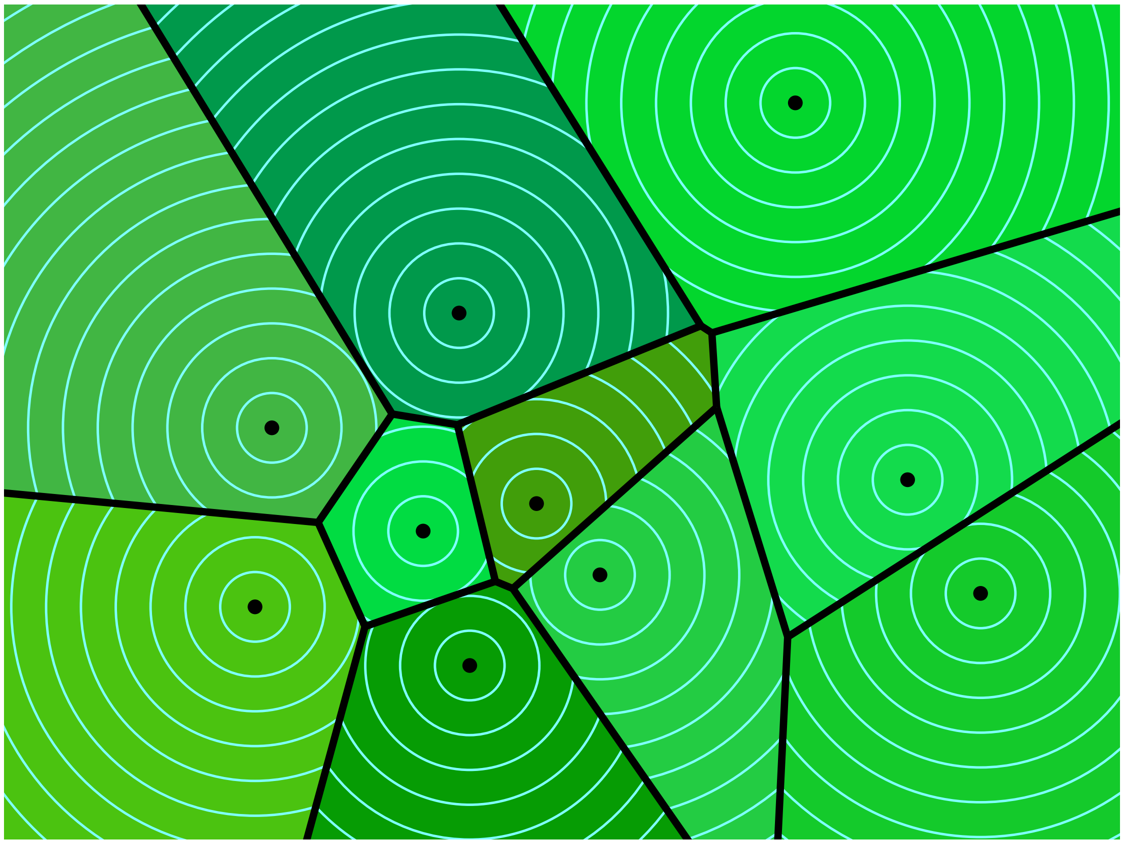
distance from v to p
as measured by v



Distance Function Examples

1. Standard Voronoi diagram

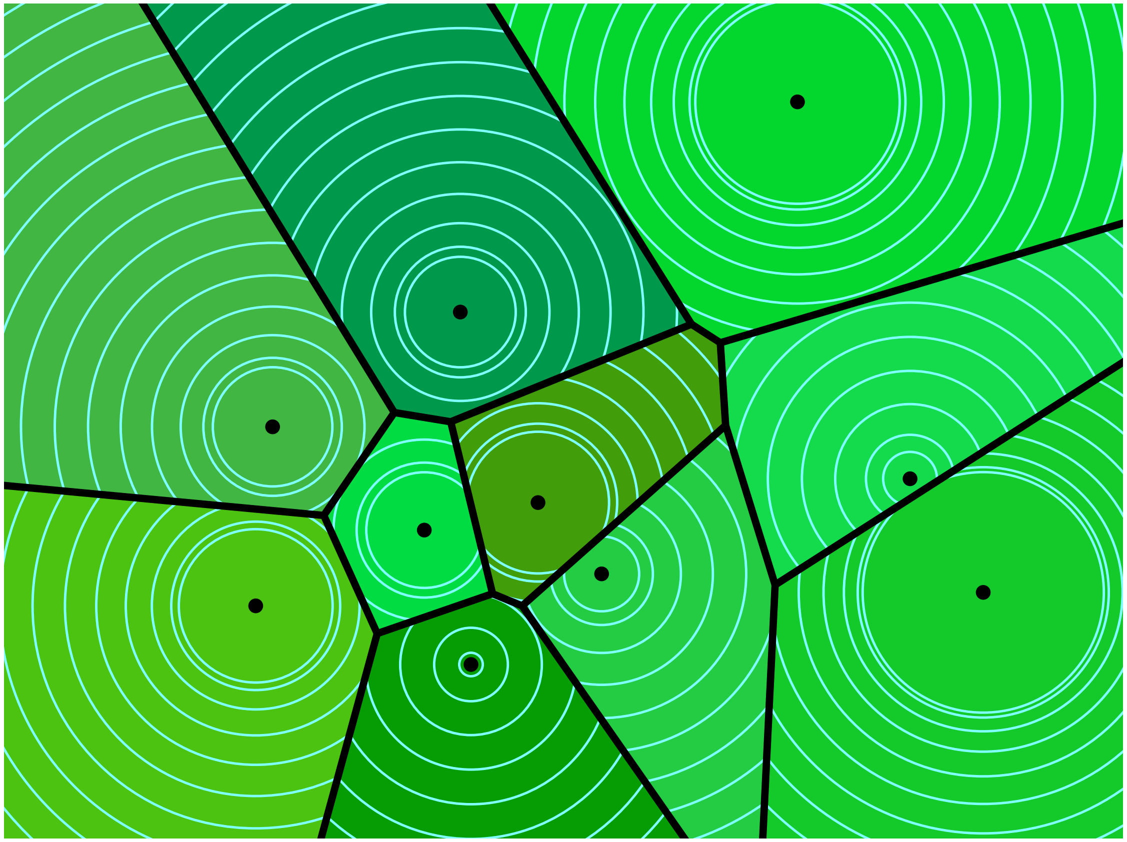
$$d_v(p) = ||p - v||_2$$



Distance Function Examples

2. Power diagram

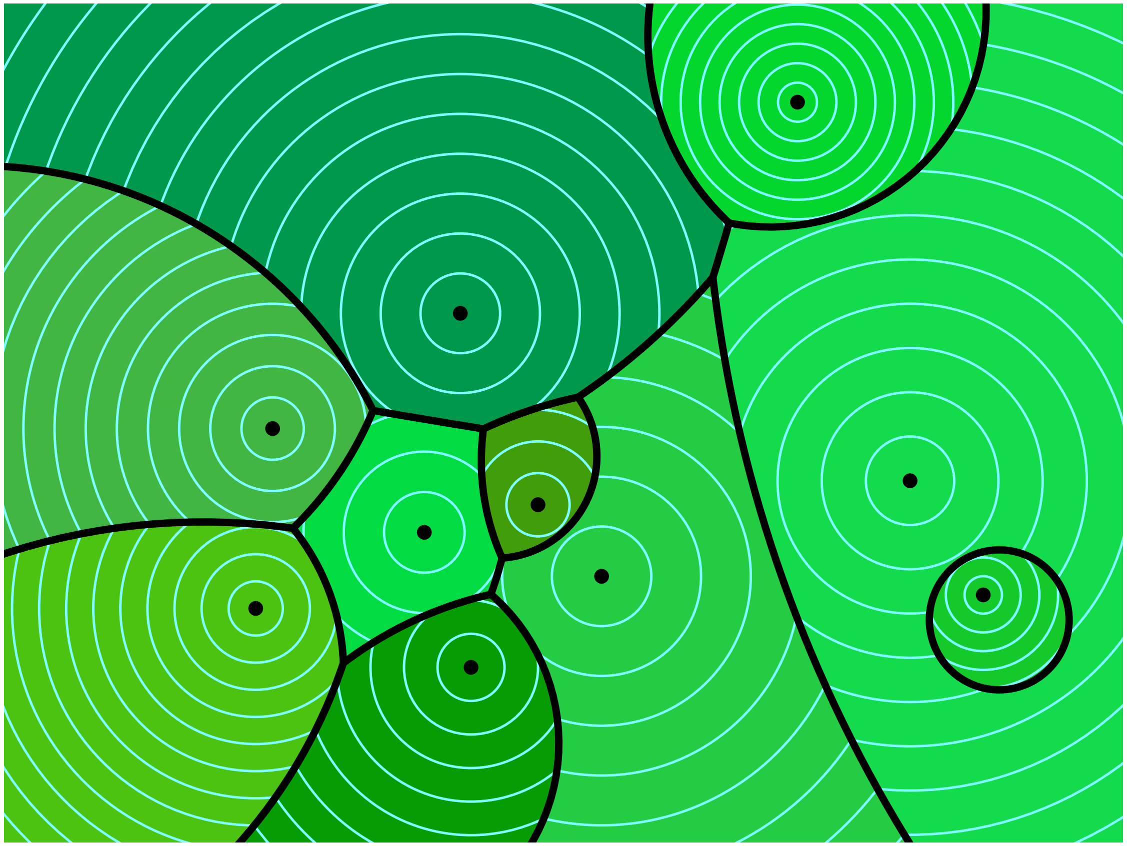
$$d_v(p) = (\|p - v\|_2^2 - c_v)^{1/2}$$



Distance Function Examples

3. Multiplicatively weighted Voronoi diagram

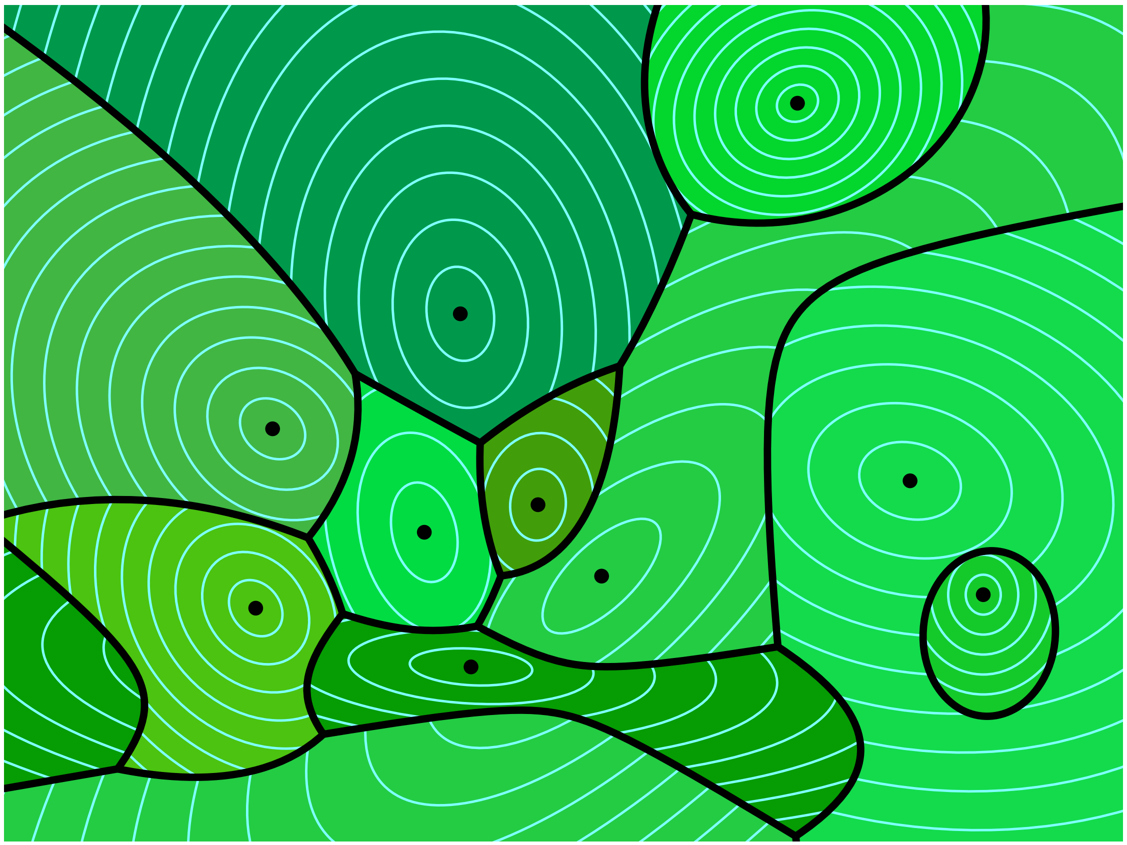
$$d_v(p) = c_v \|p - v\|_2$$



Distance Function Examples

4. Anisotropic Voronoi diagram

$$d_v(p) = [(p - v)^T M_v (p - v)]^{1/2}$$

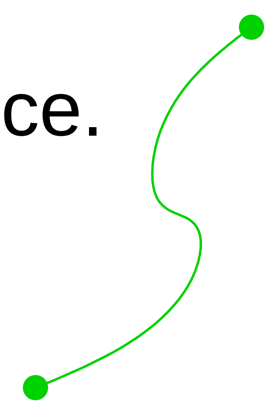


Distance Function Examples

5. Riemannian Voronoi diagram

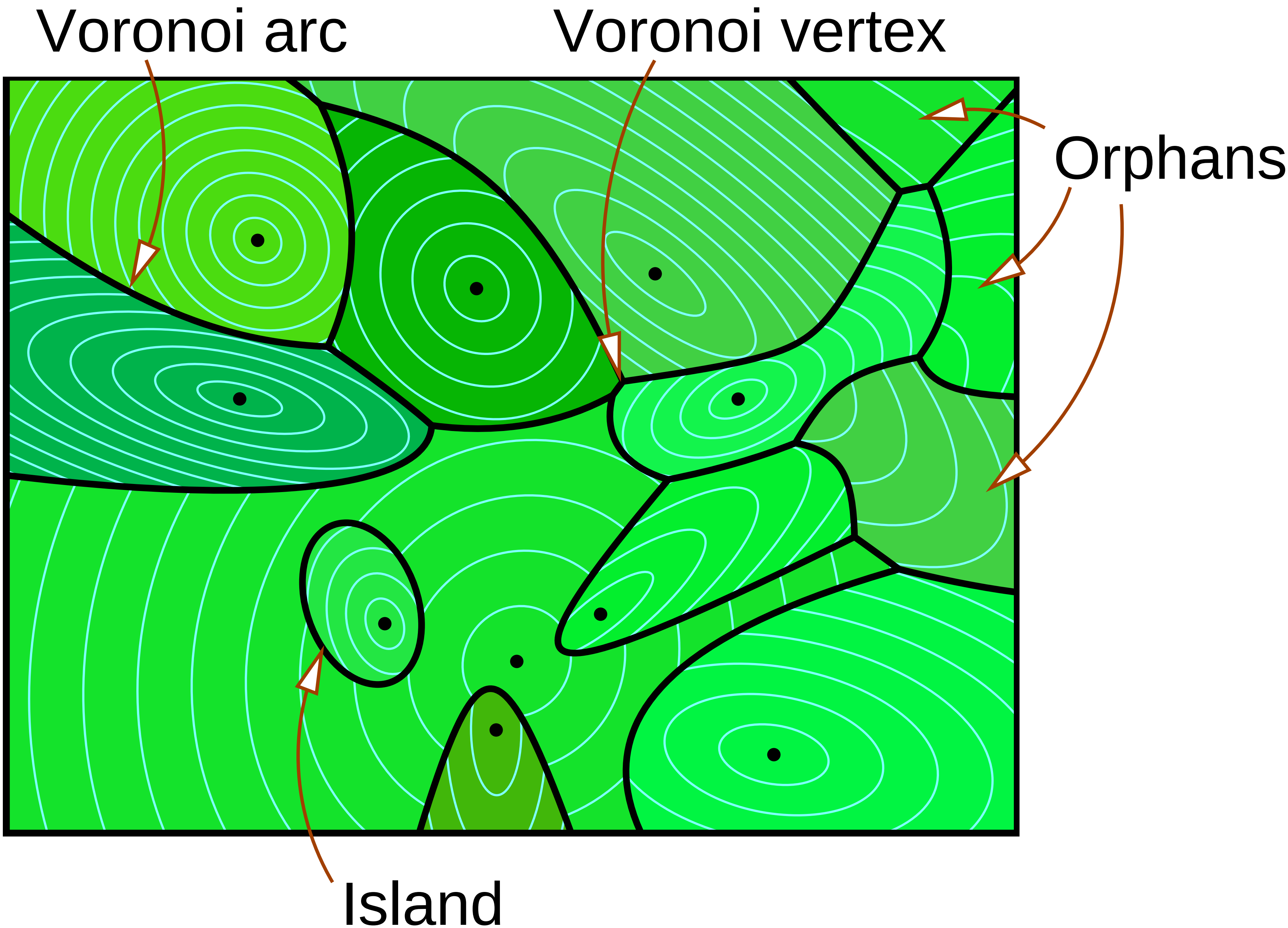
$d_v(p)$ = shortest geodesic path between v and p.

- Leibon & Letscher [2000] define Voronoi/Delaunay on Riemannian manifolds.
  - Bounded curvature + densely sampled sites → well−defined Delaunay triangulation.
  - Geodesics too hard to compute in practice.



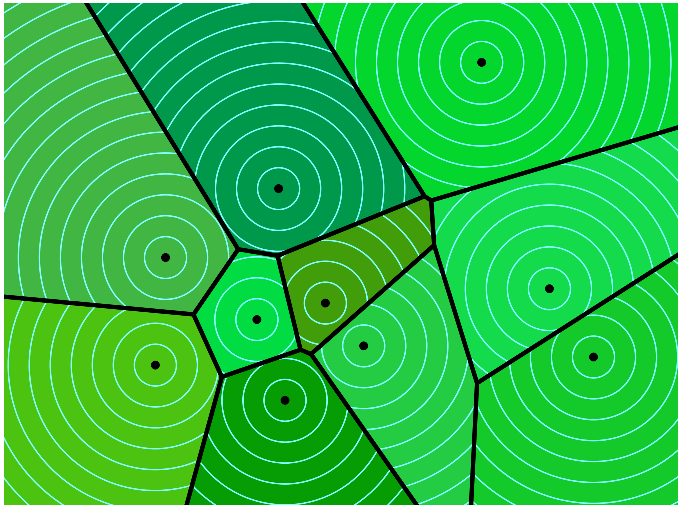
- George & Borouchaki [1998] suggest fast heuristic approximation to Riemannian Delaunay, but can’t prove anything.

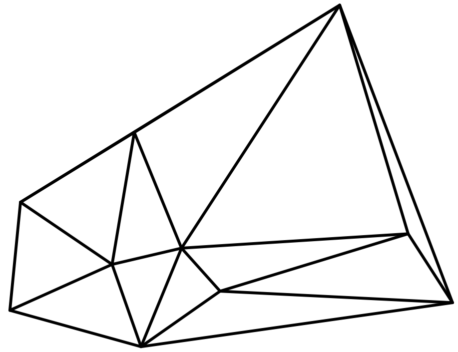
Anisotropic Voronoi Diagram



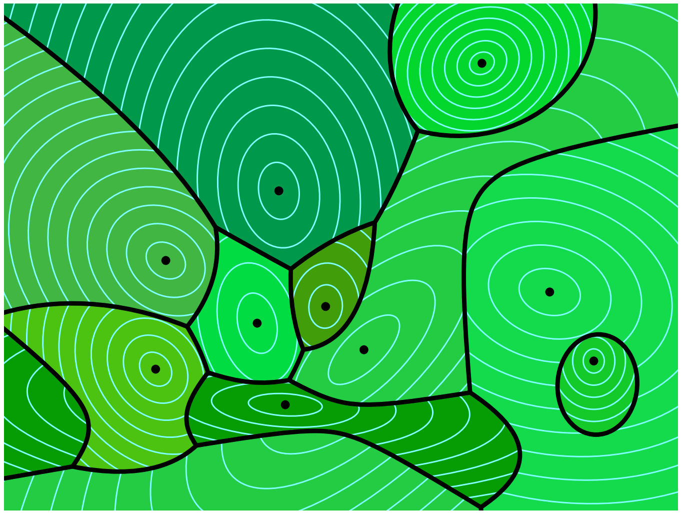
Duality

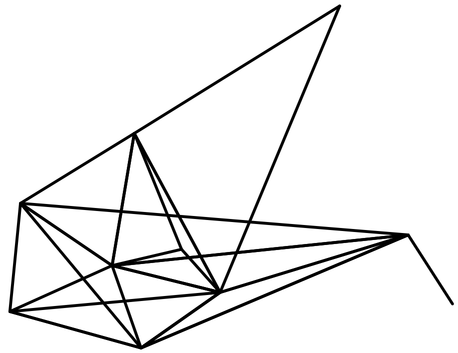
The dual of the standard Voronoi diagram is the Delaunay triangulation.





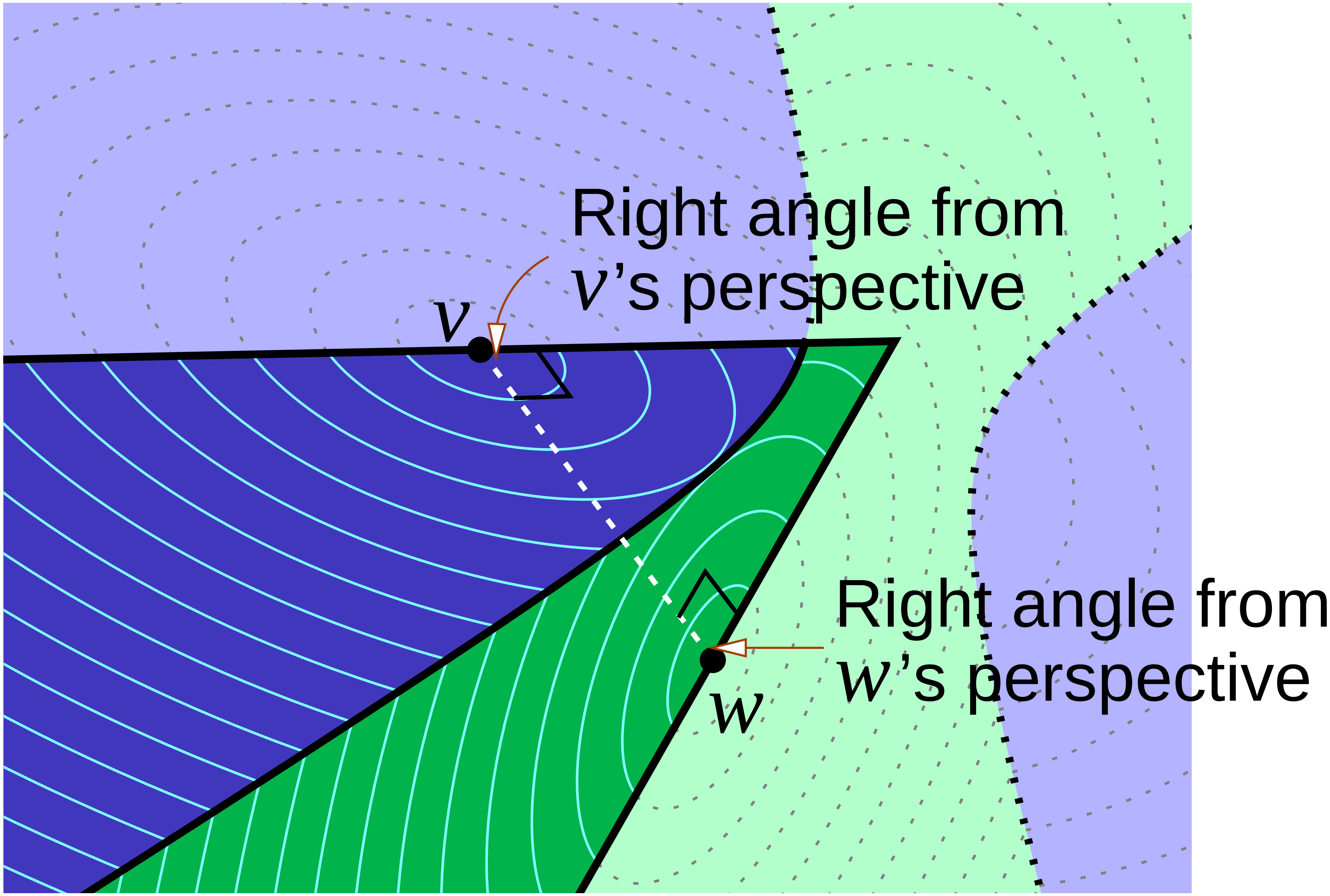
The dual of the anisotropic Voronoi diagram is not, in general, a triangulation.



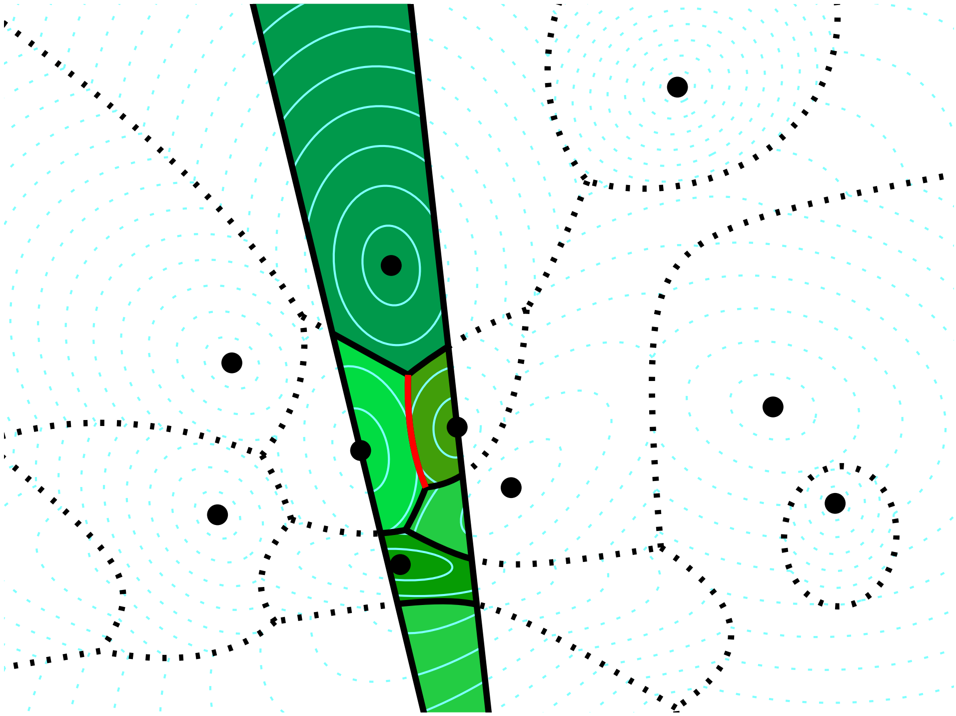


We must enforce some extra conditions.

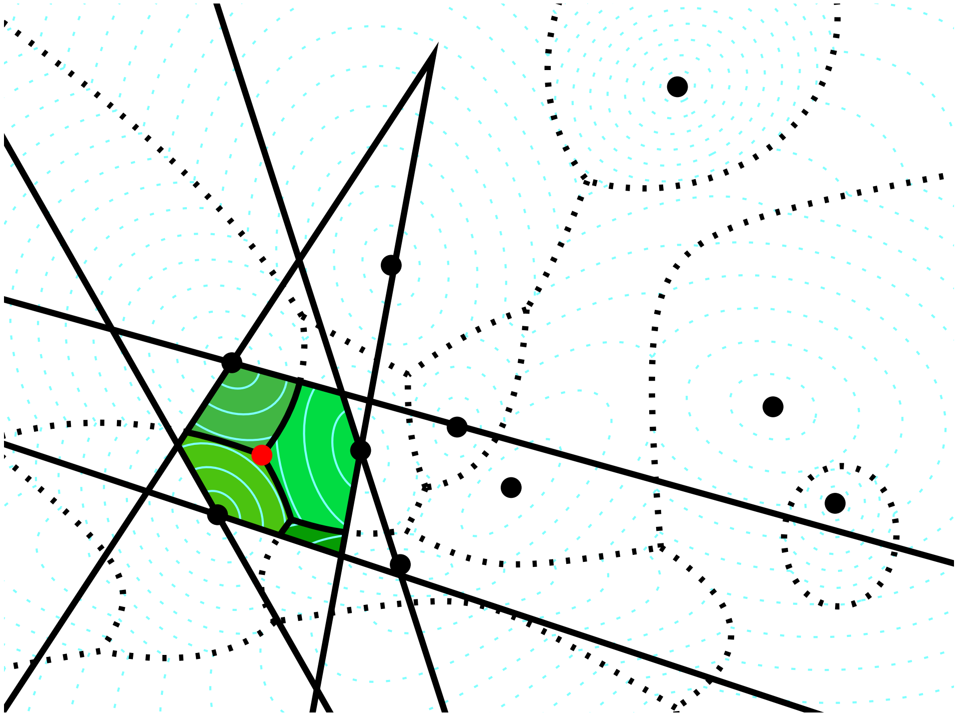
Two Sites Define a *Wedge*



Voronoi arc is *wedged* if it's in the wedge of the sites that define it.

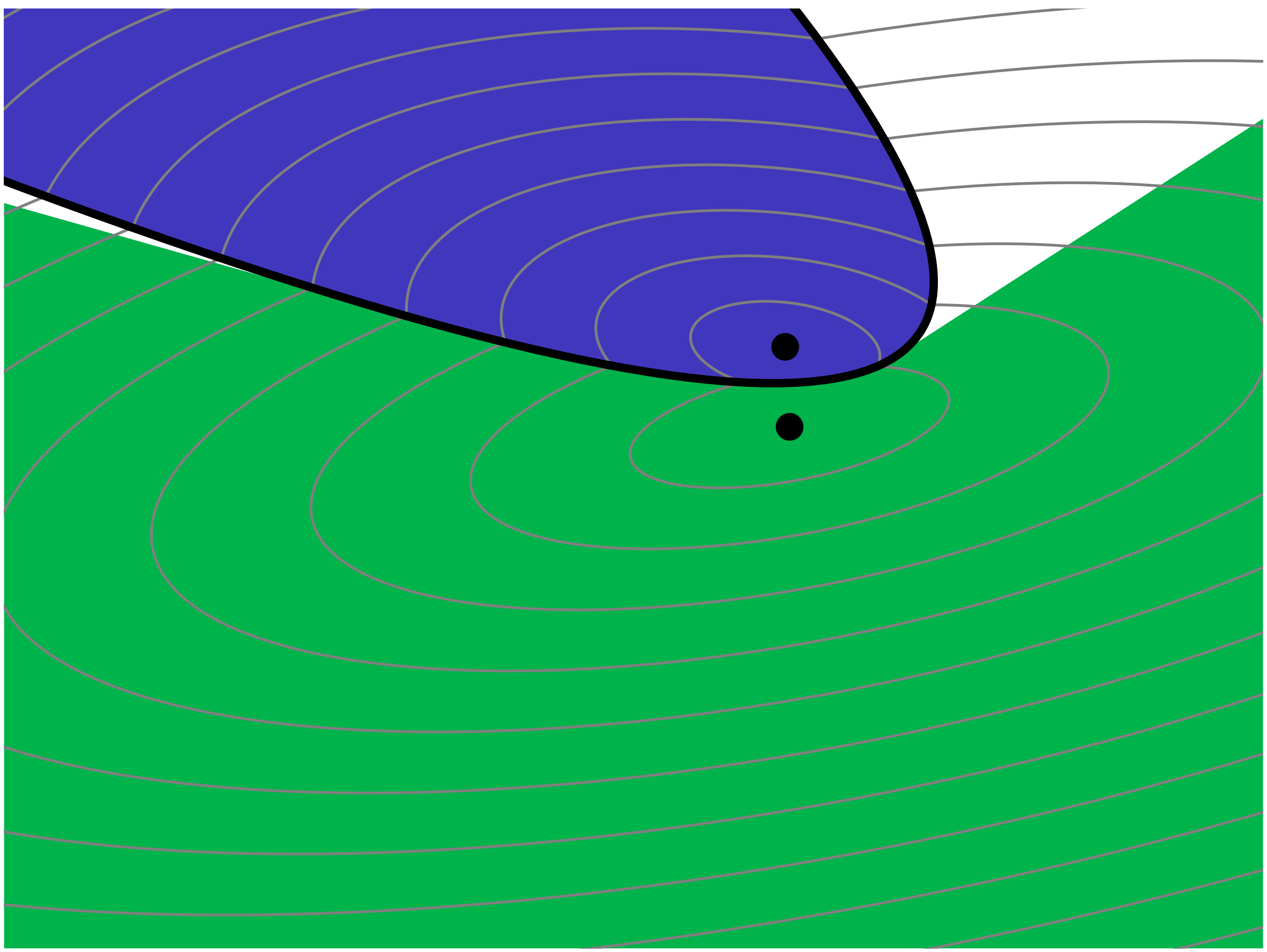


Voronoi vertex is *wedged* if it's in all 3 wedges.



Visibility Lemma

Inside wedge, each site sees its whole Voronoi cell.



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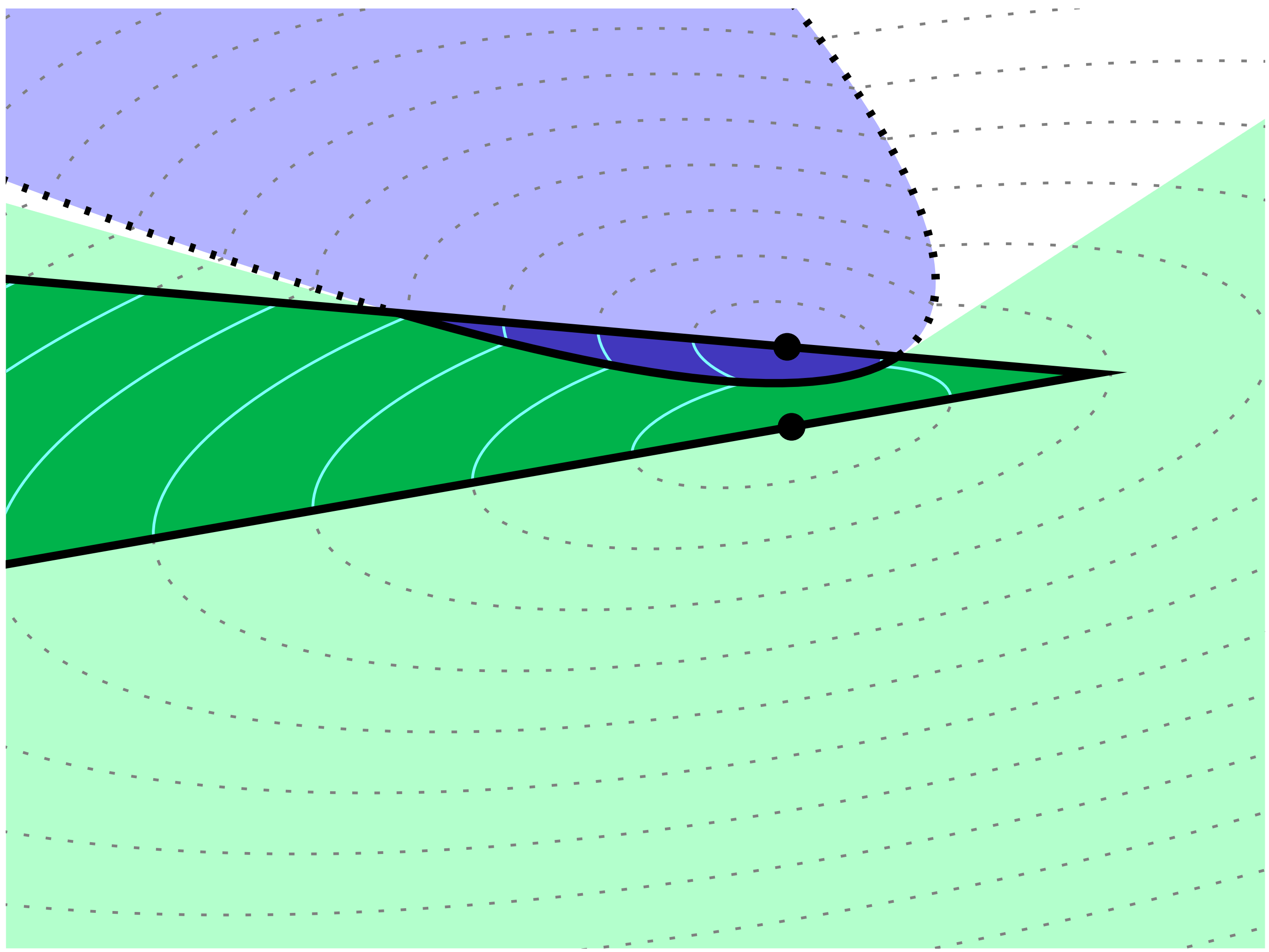
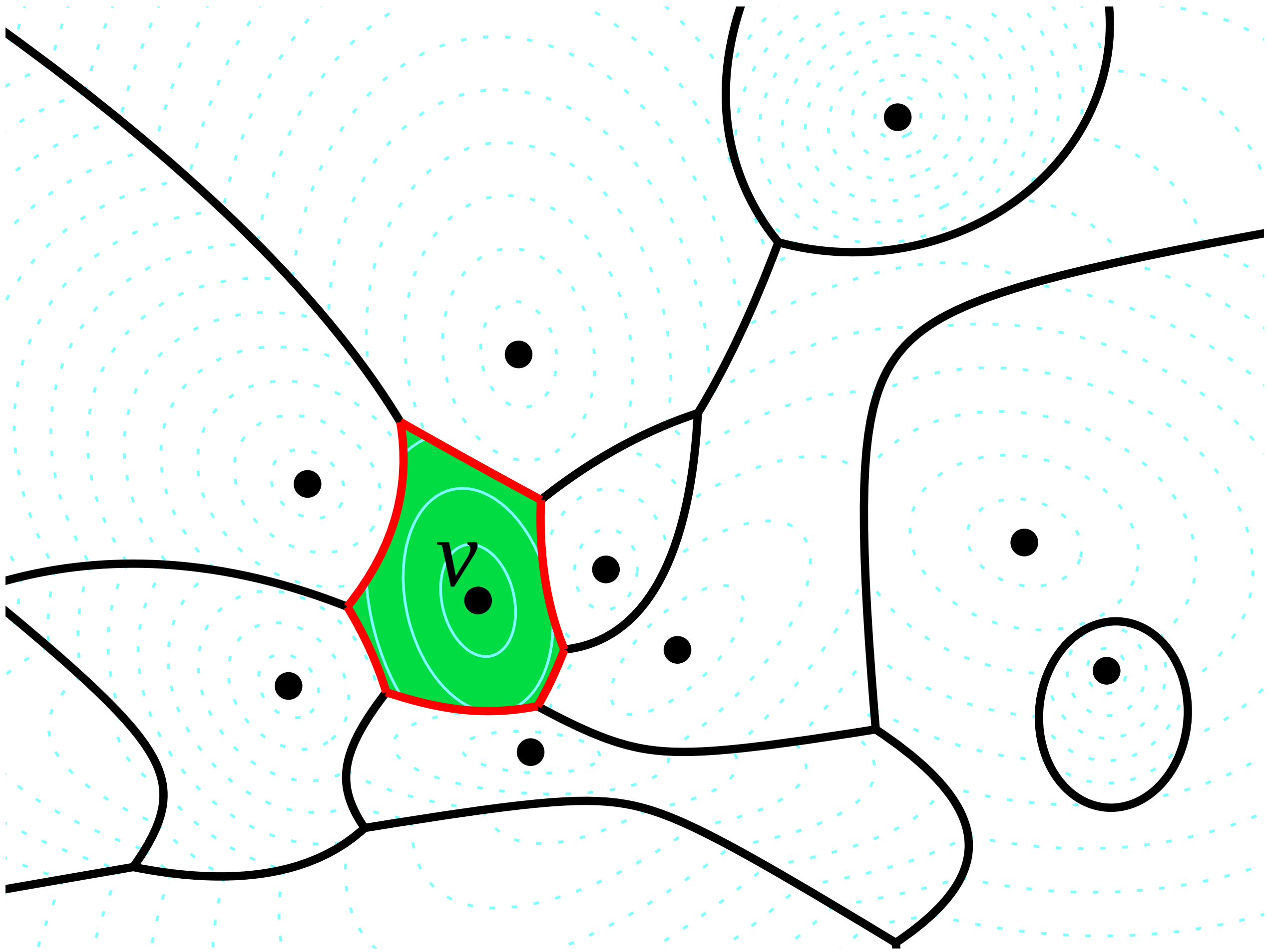


Visibility Theorem

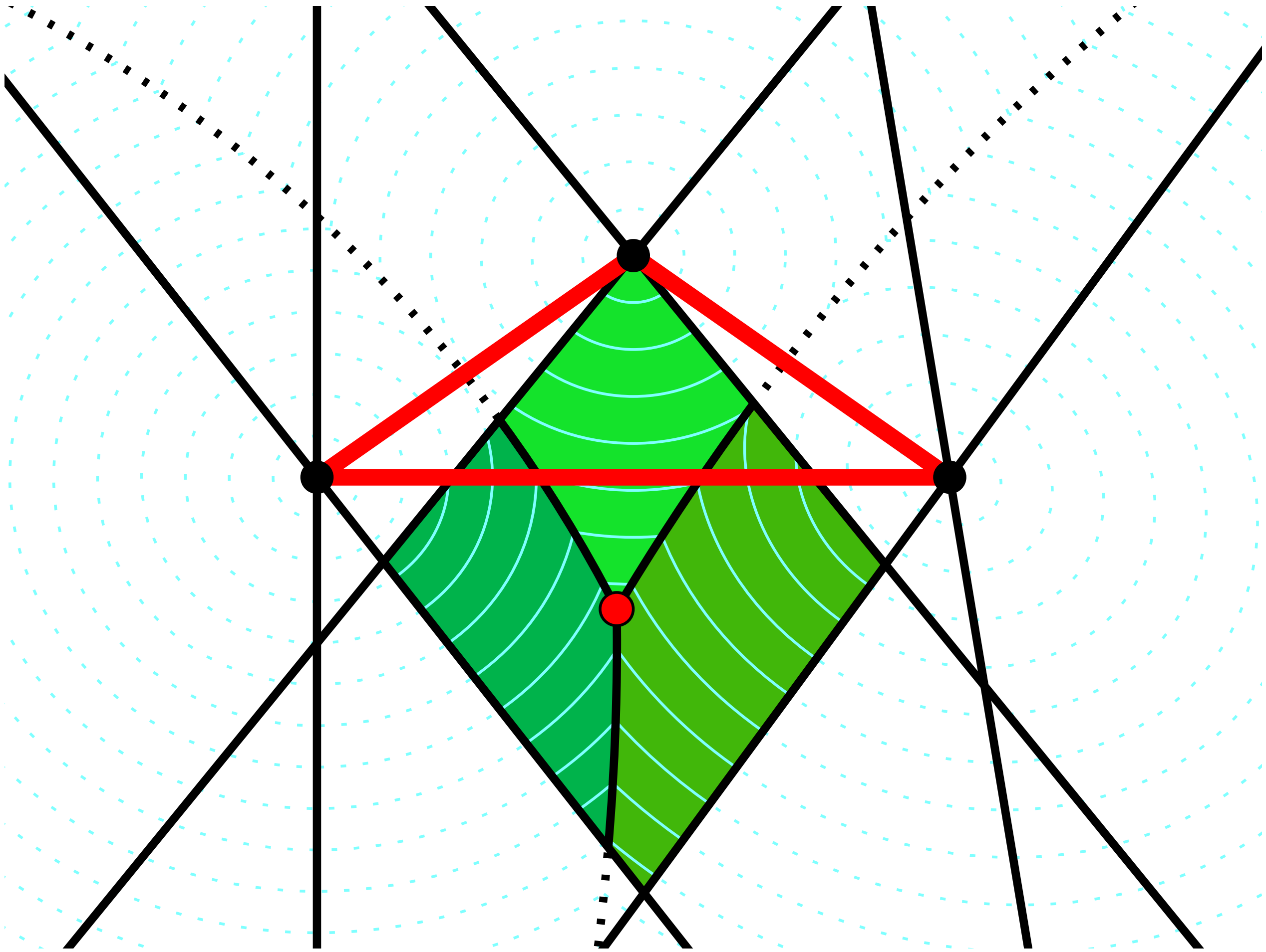
If every Voronoi arc of Vor(v) is wedged, then Vor(v) is star−shaped & visible from v.



(This generalizes to higher dimensions.)

Triangle Orientation Lemma

If a Voronoi vertex is wedged, its dual triangle has positive orientation.

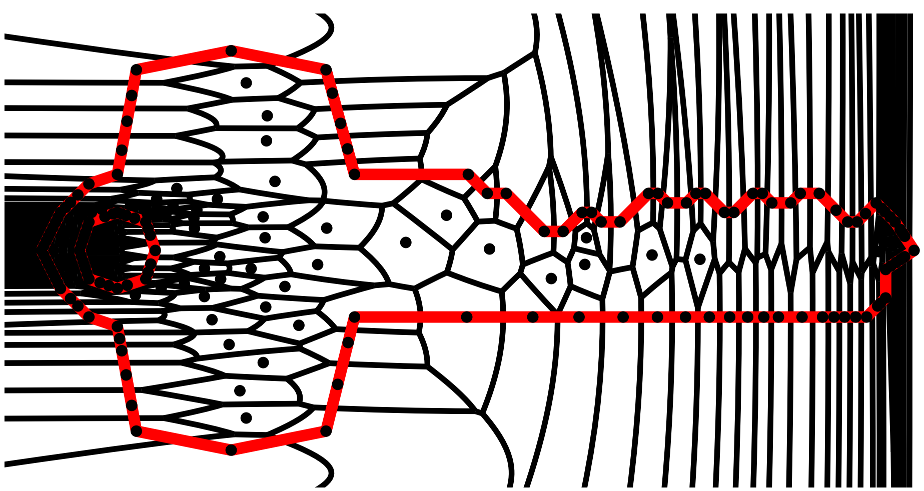


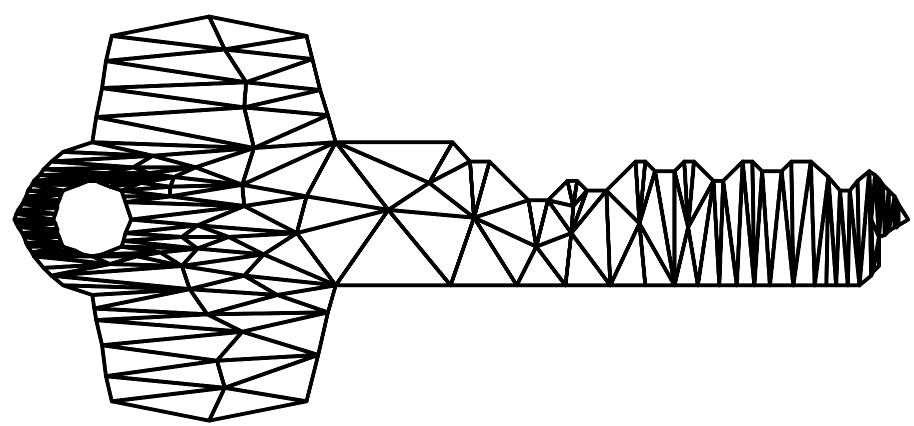
(Does not generalize above two dimensions.)

Dual Triangulation Theorem

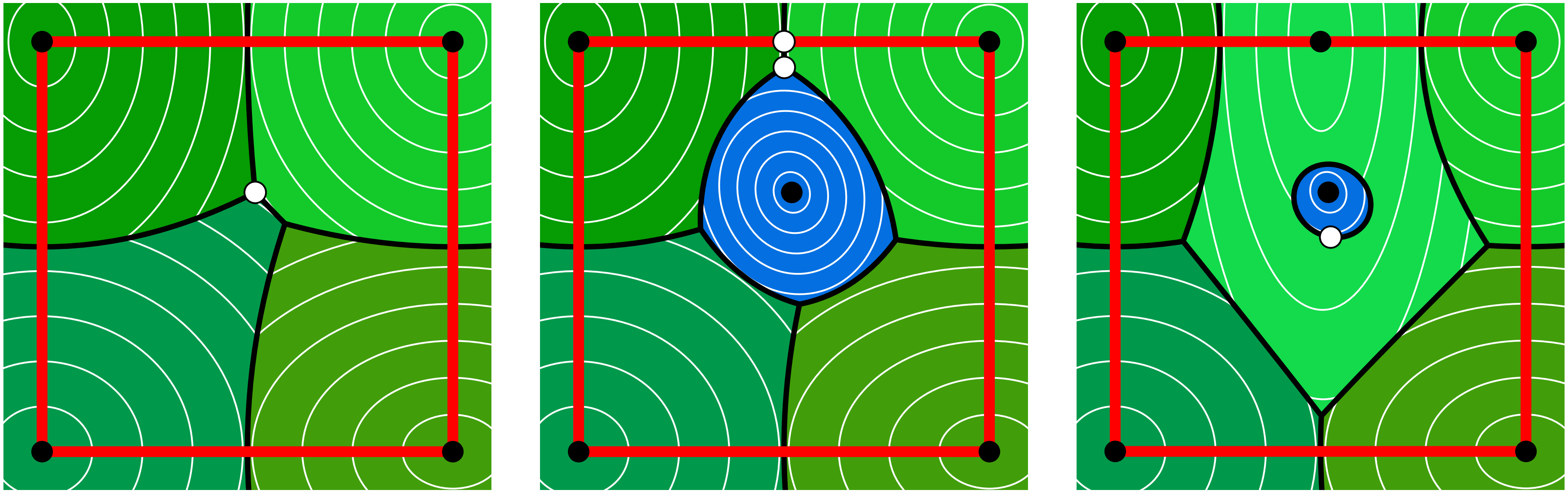
If all arcs & vertices are wedged, Voronoi diagram dualizes to an *anisotropic Delaunay triangulation.*

If arcs & vertices are wedged *inside a domain* (& some conditions hold at the boundary), the dual is a triangulation of the domain.



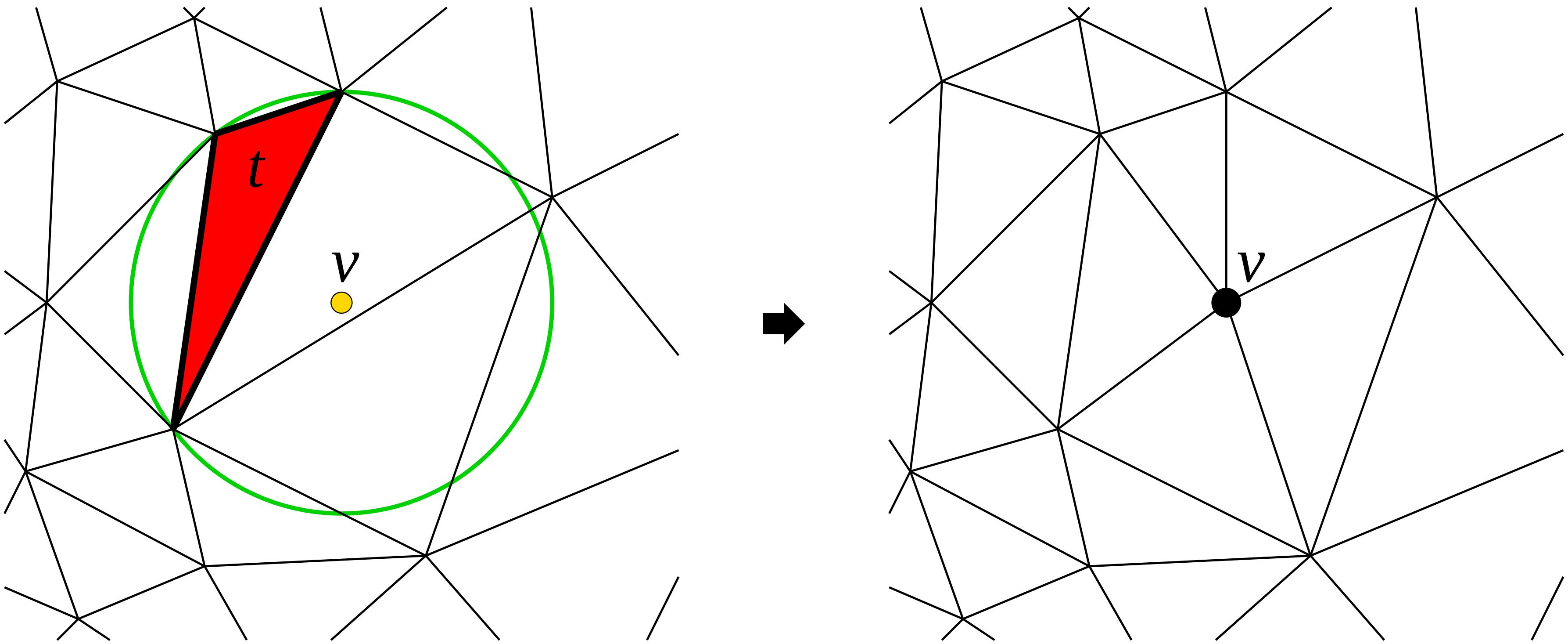


III. Anisotropic Mesh Generation by Voronoi Refinement



Isotropic Mesh Generation by Delaunay Refinement

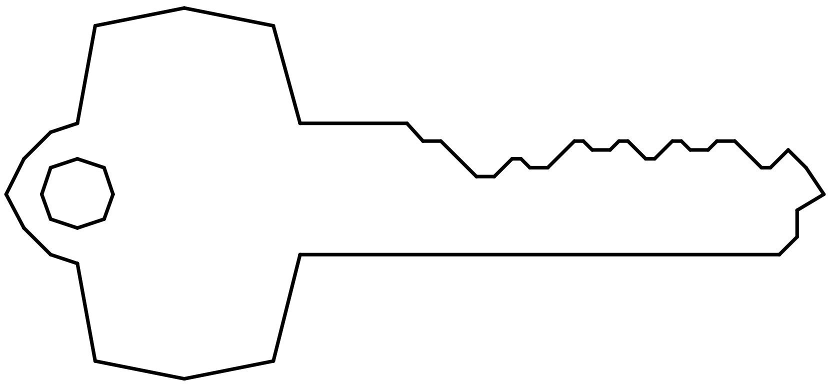
(William Frey, L. Paul Chew, Jim Ruppert)

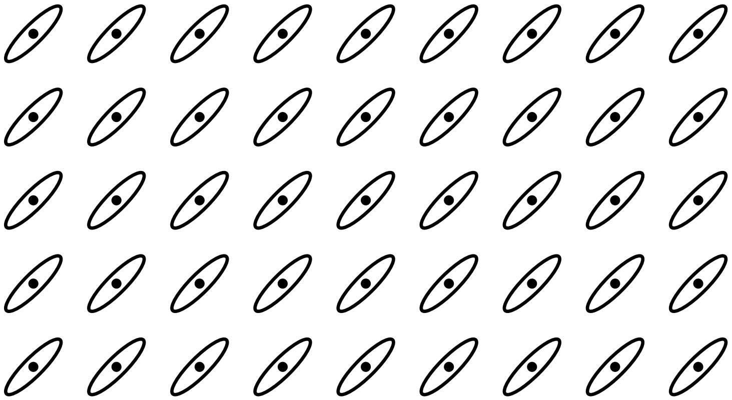


- Always maintain Delaunay triangulation.
- Eliminate any triangle with small angle (< 20°) by inserting vertex at center of circumscribing circle.
- No smaller edge is introduced → guaranteed to terminate.

This solves the isotropic case, M= identity.

Easy Case: M = constant





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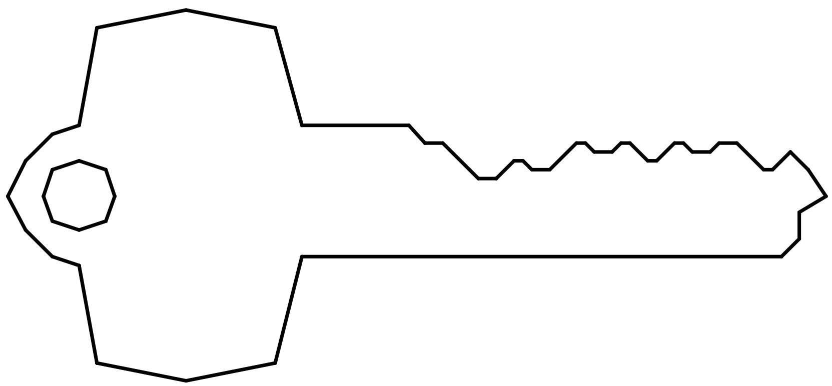
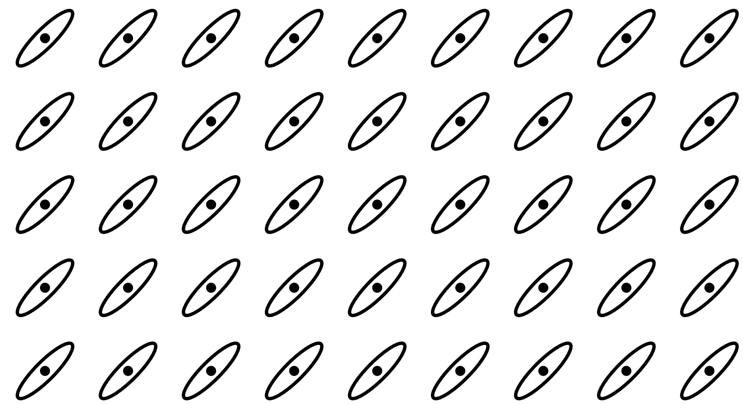
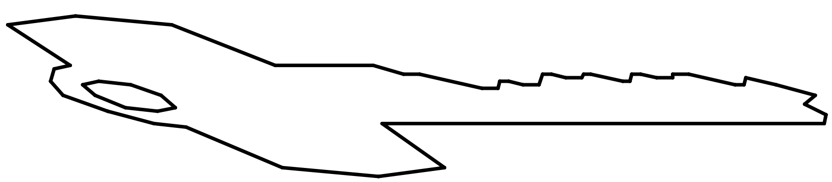
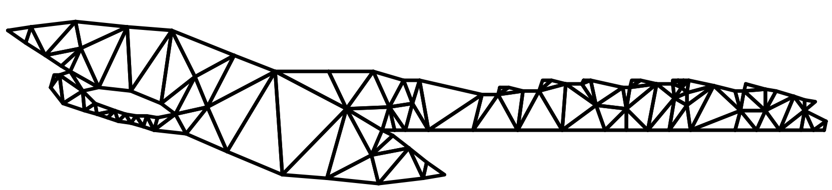




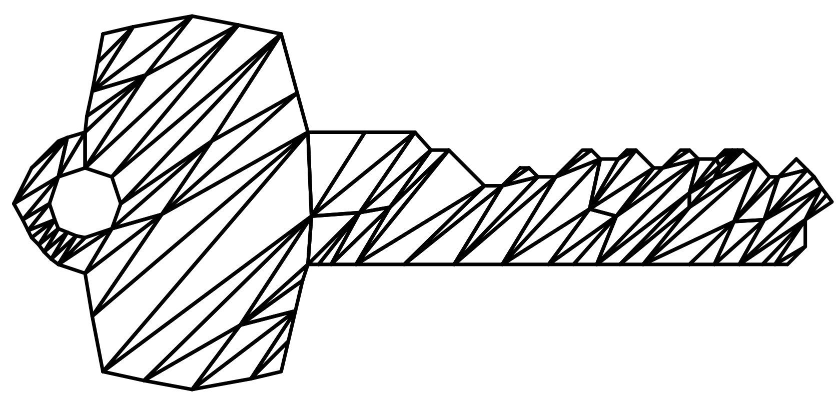
1. Apply F to the domain



2. Isotropic meshing

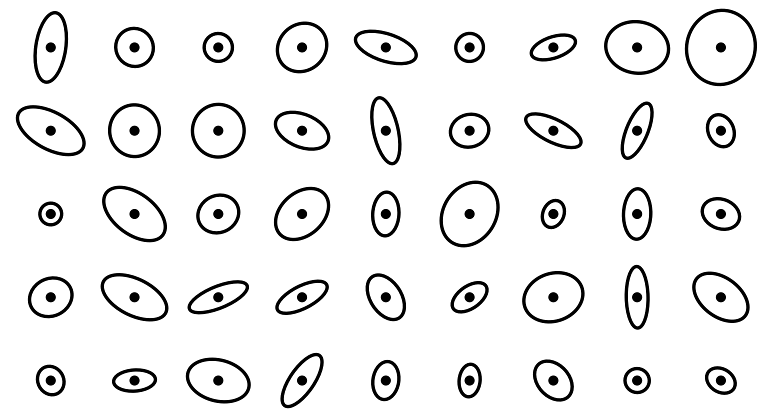


3. Apply F^{-1}



Remarks on Anisotropy

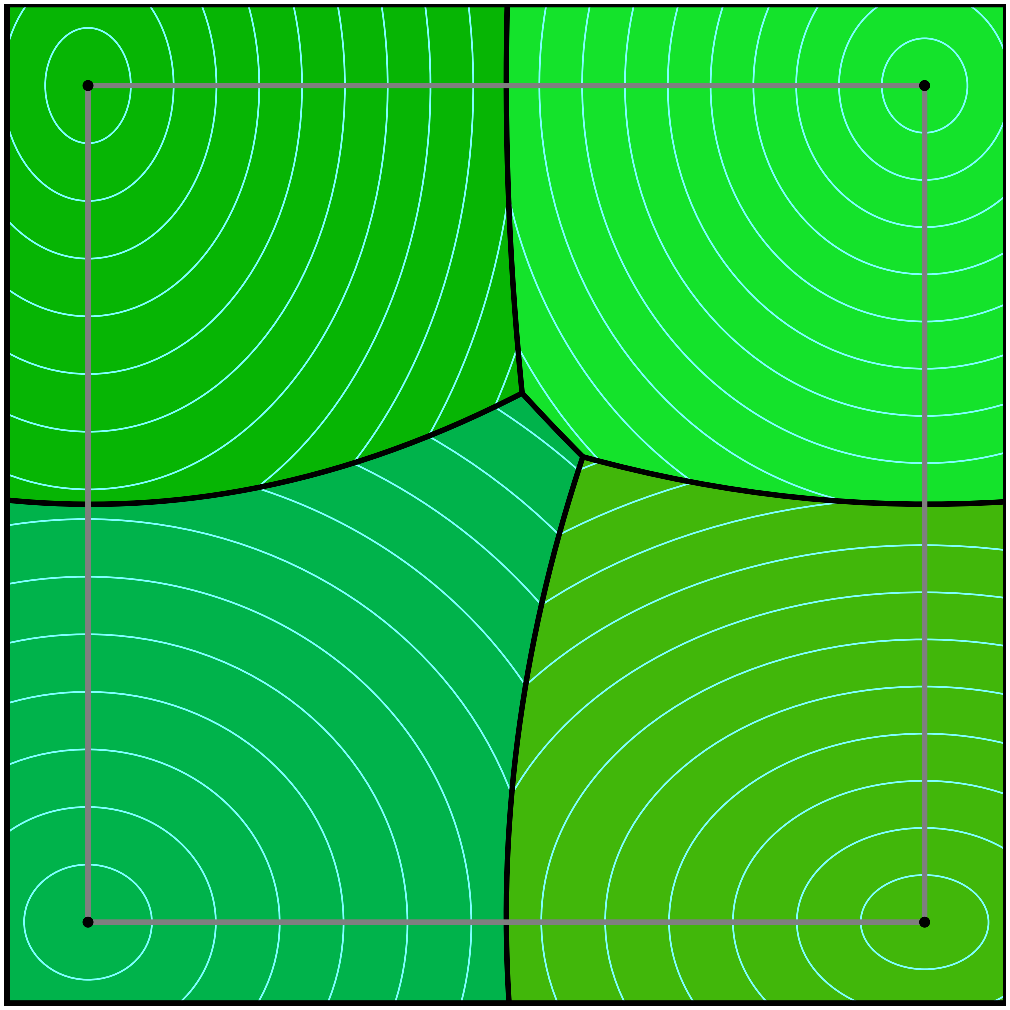
- Large distortion isn't a problem.
- *Rapid variation* in the metric tensor field is a problem.



About our Algorithm

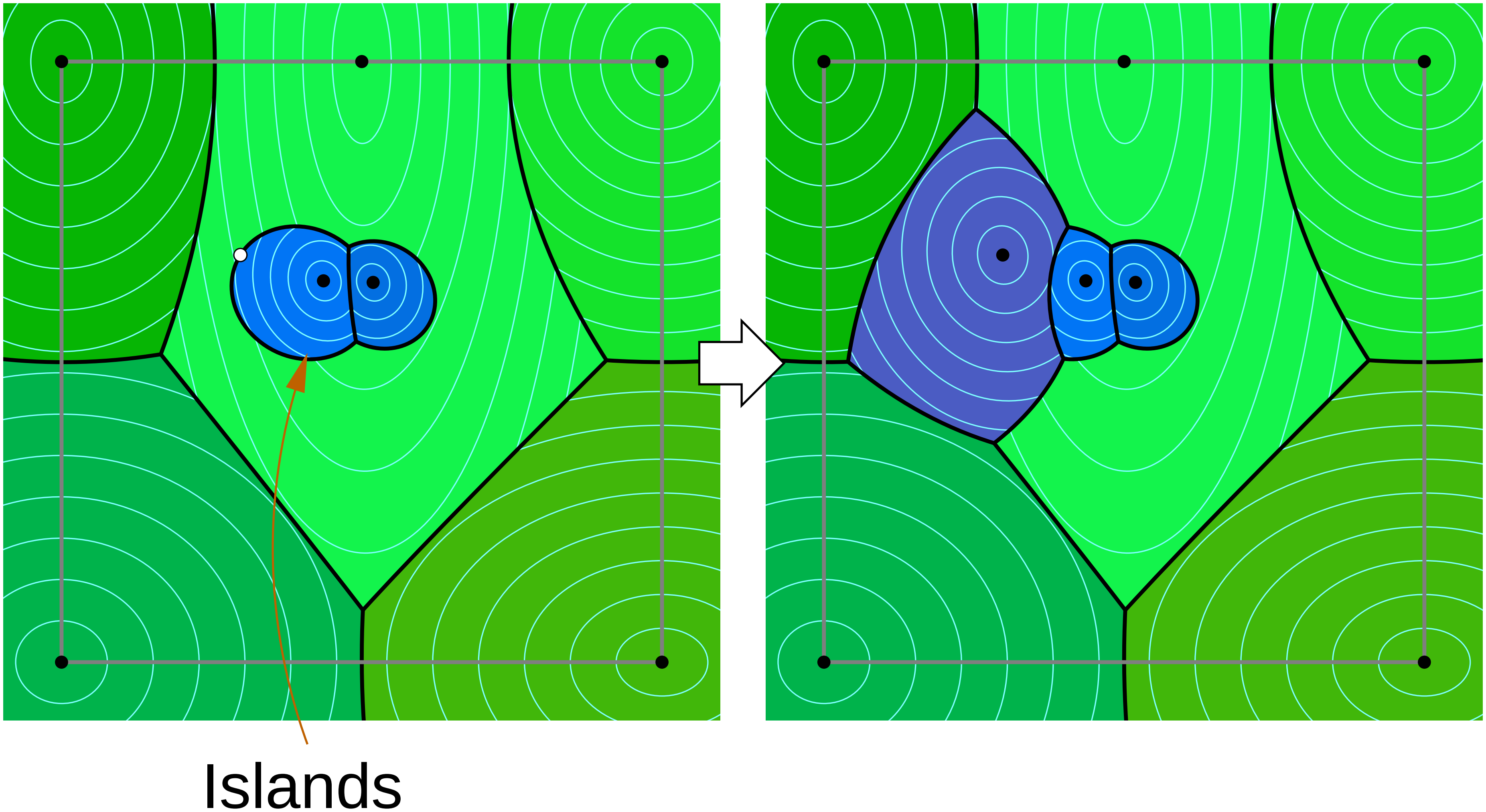
- First algorithm for *guaranteed−quality* anisotropic meshing.
- Reduces to standard Delaunay refinement when M is constant.
- We can quantify how much refinement is caused by variation in M.

Voronoi Refinement Algorithm



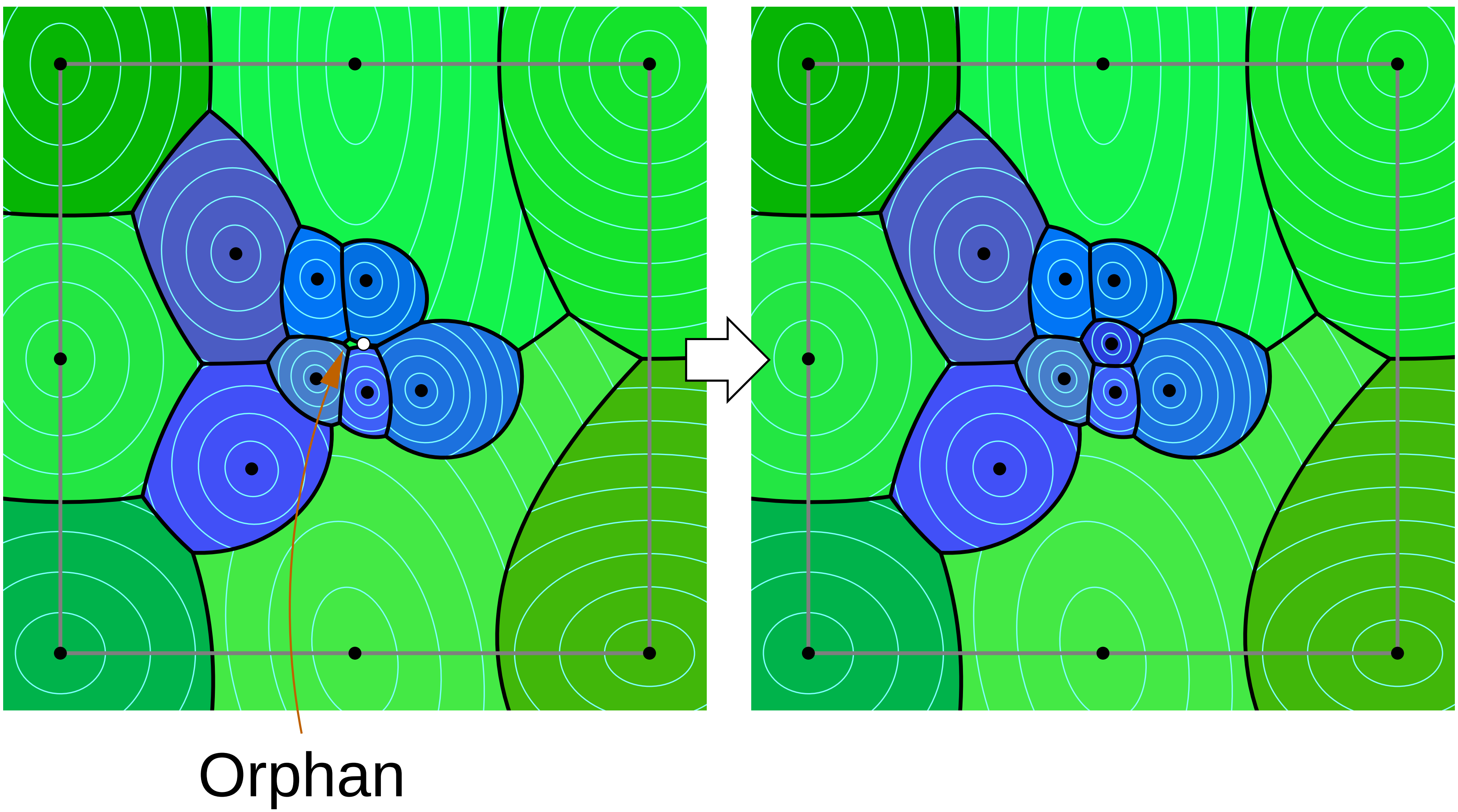
Begin with the anisotropic Voronoi diagram of the vertices of the domain.

Voronoi Refinement Algorithm



Insert new sites on unwedged portions of arcs.

Voronoi Refinement Algorithm



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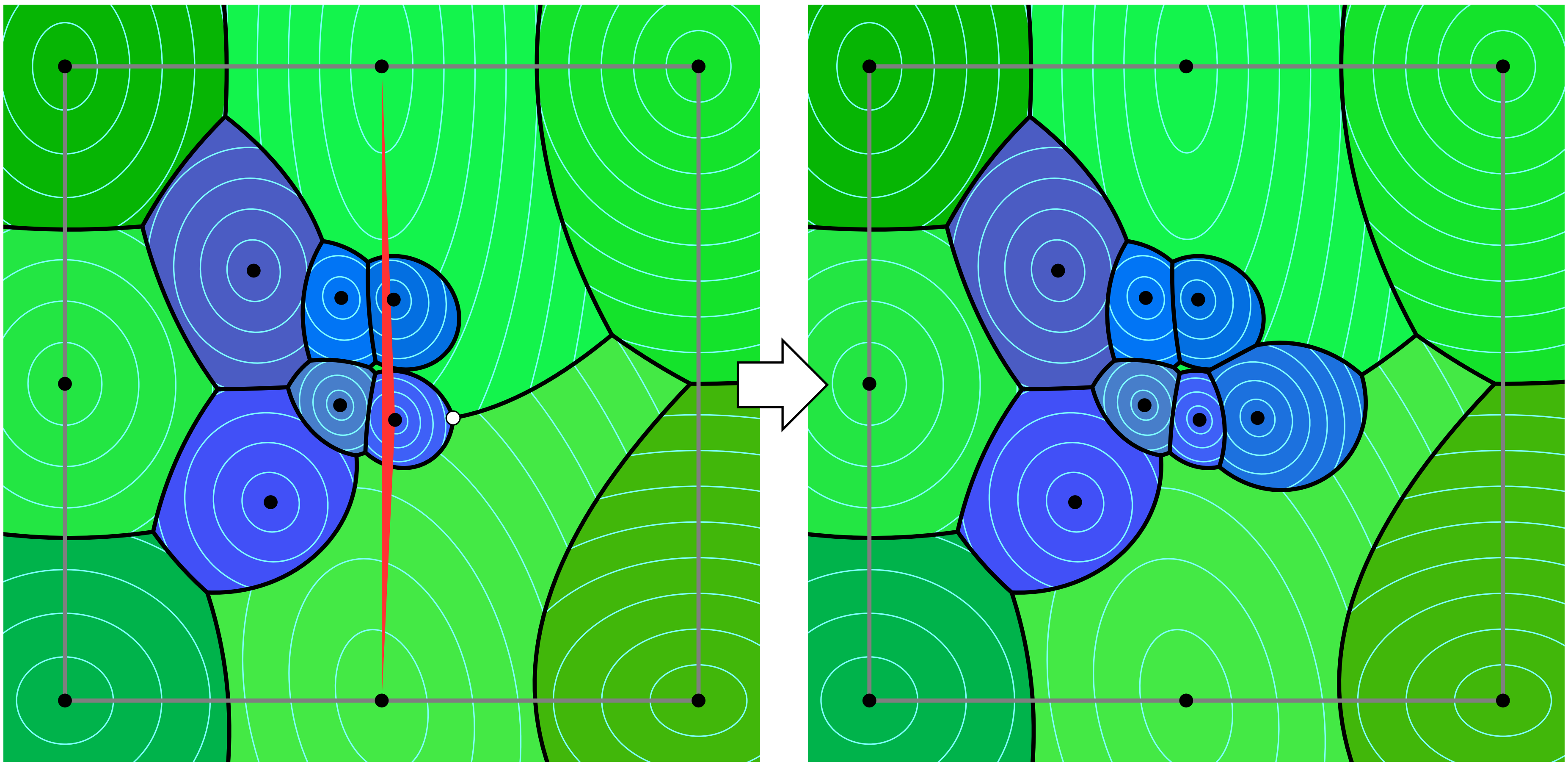
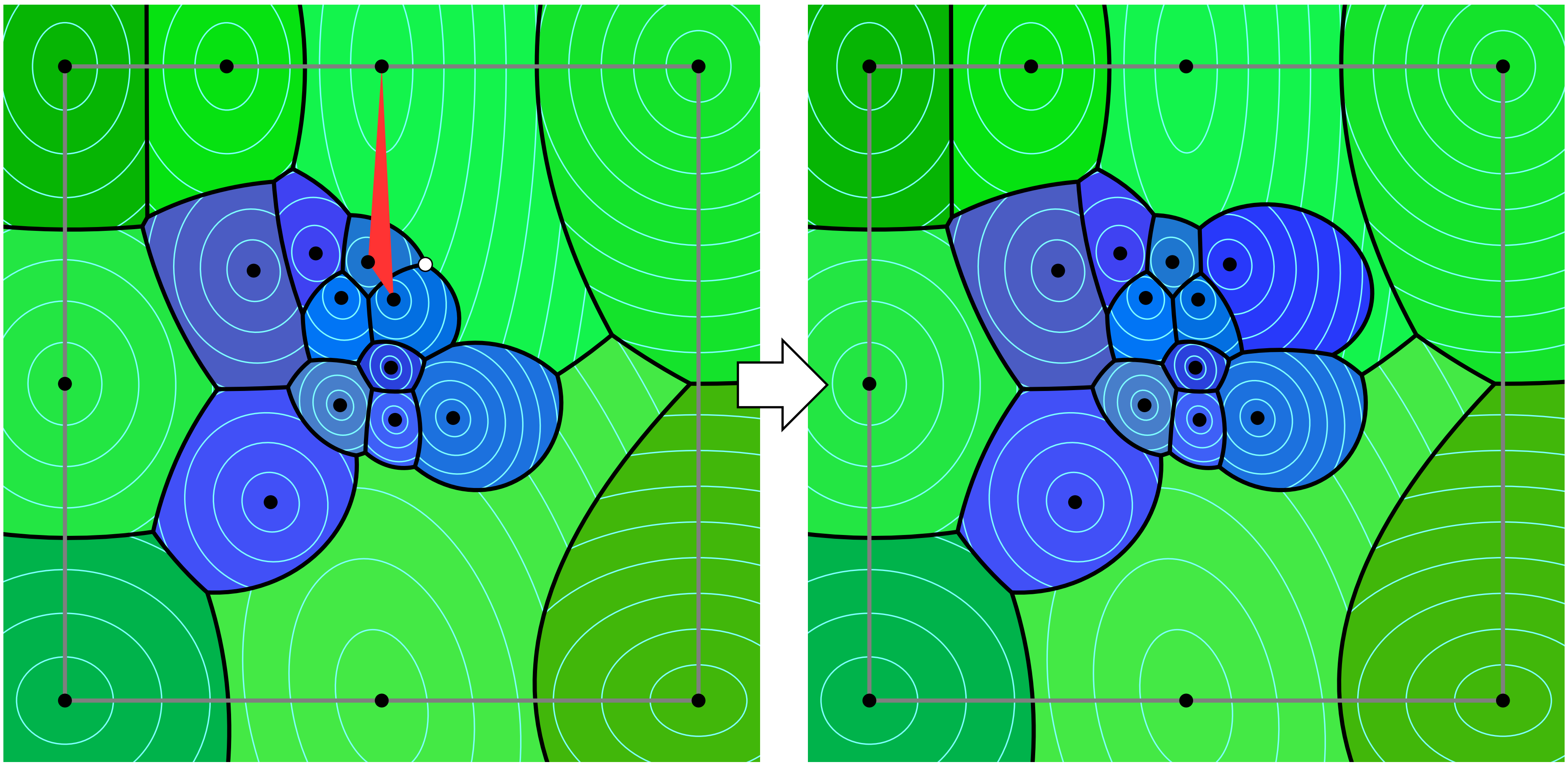


Insert new sites at Voronoi vertices
that dualize to inverted triangles.

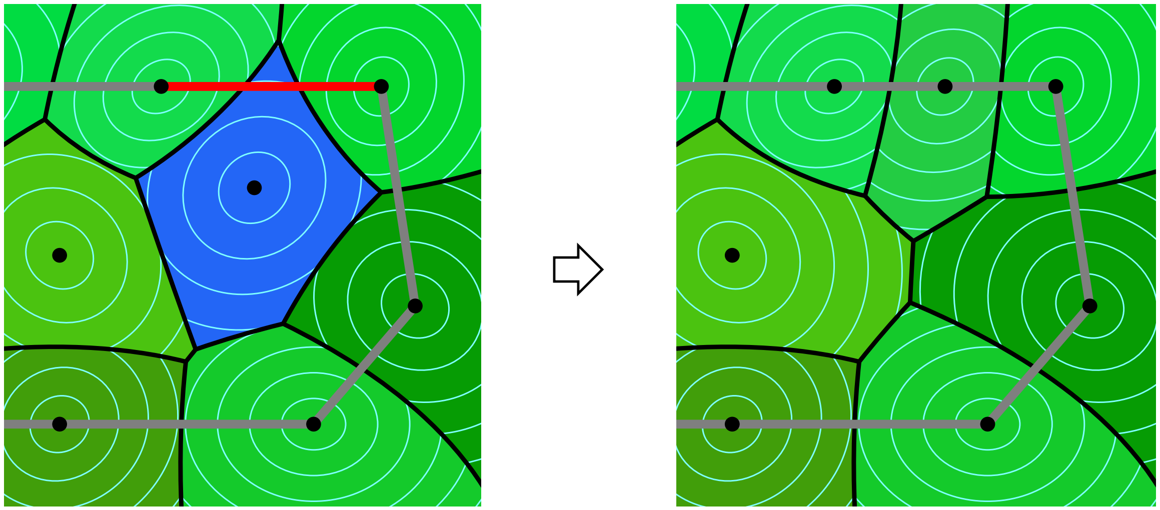
Voronoi Refinement Algorithm



Insert new sites at Voronoi vertices
that dualize to poor−quality triangles.

Special Rules for the Boundary

Encroachment:
a segment is split
if it intersects
a cell not
belonging to
an endpoint.



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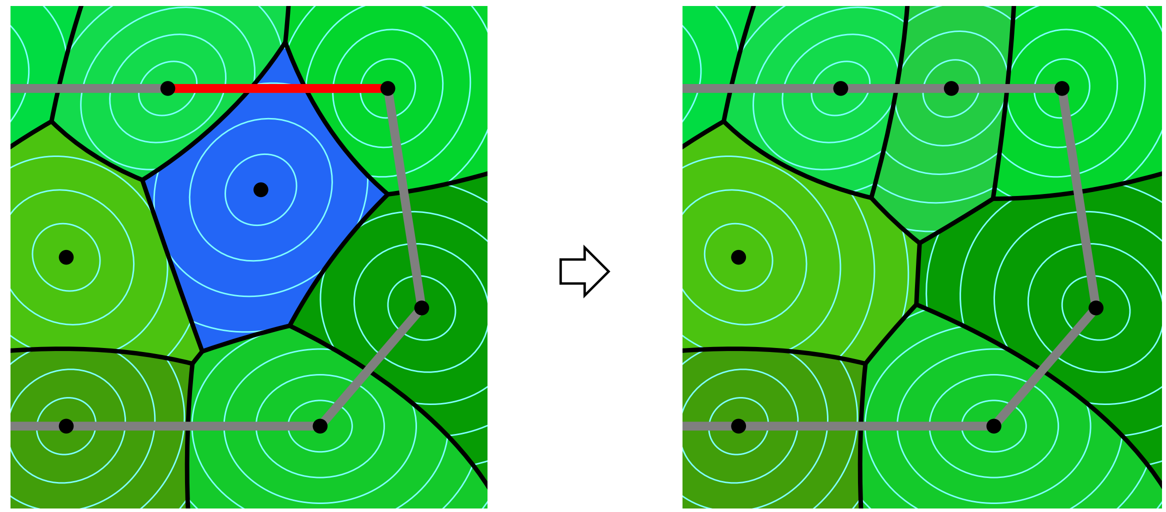
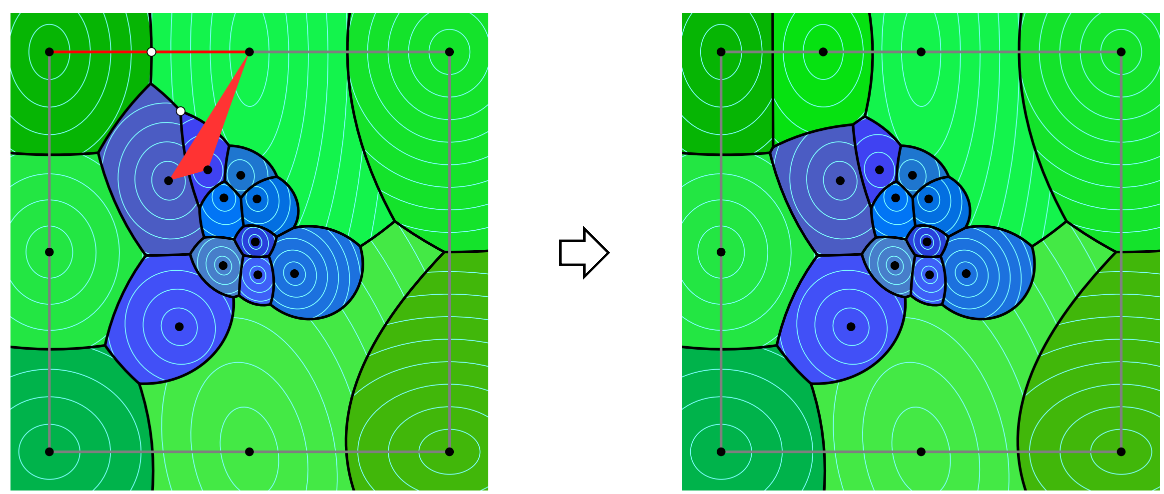
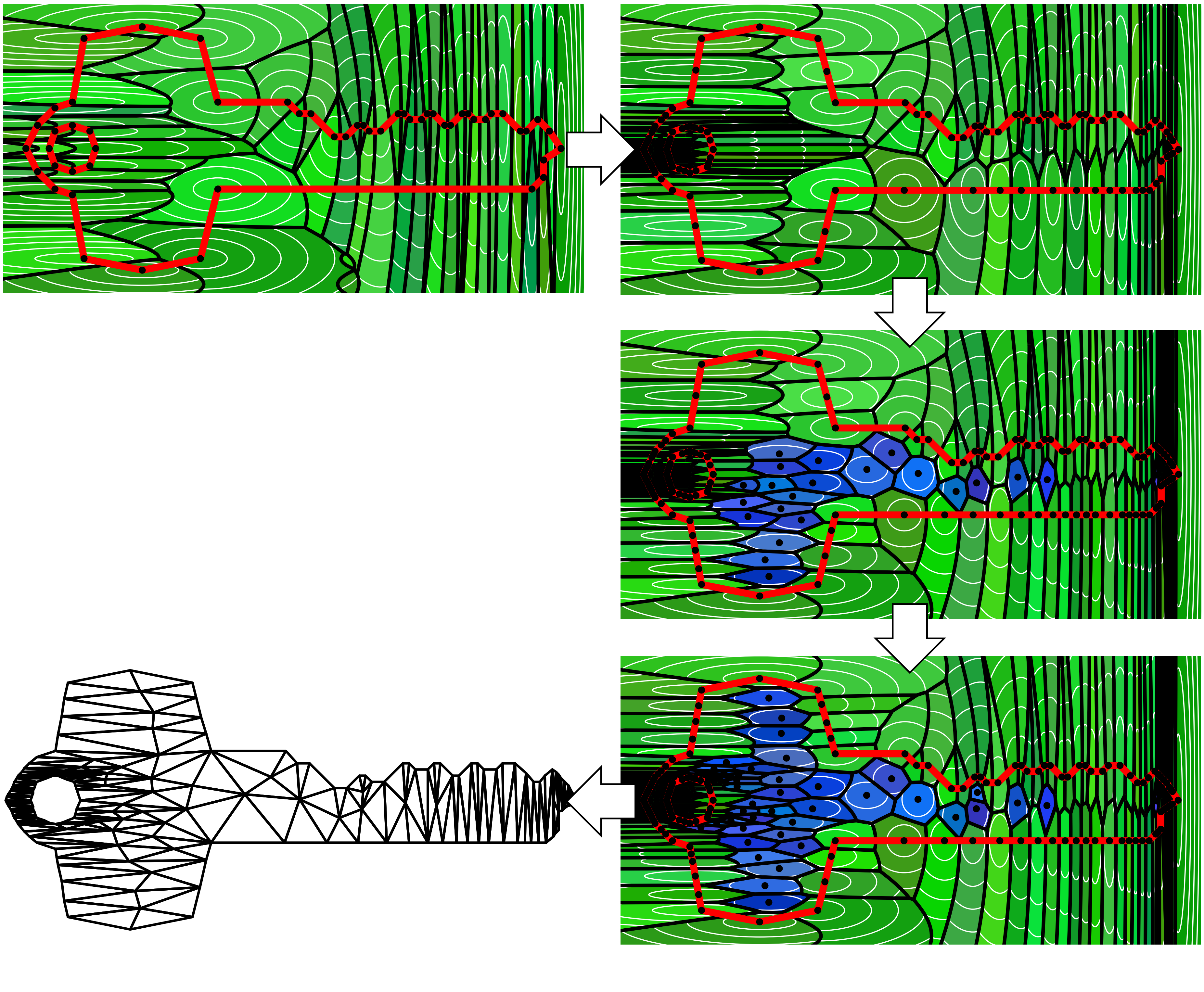


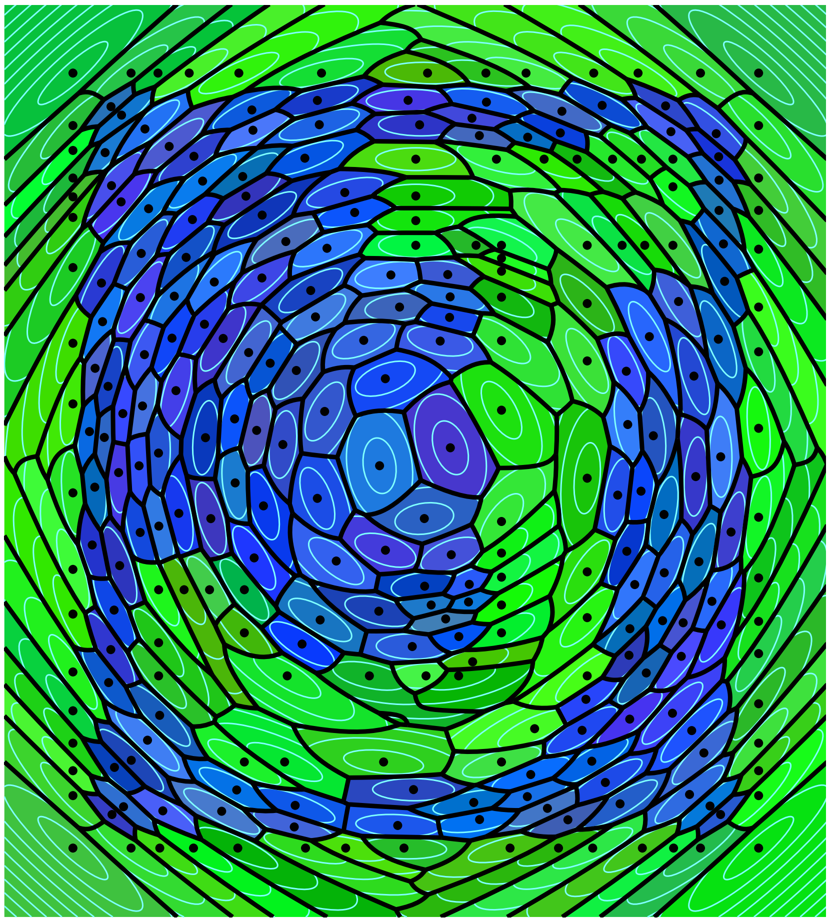
Insertion of
encroaching sites
is (usually)
forbidden.
Split the
segment instead.

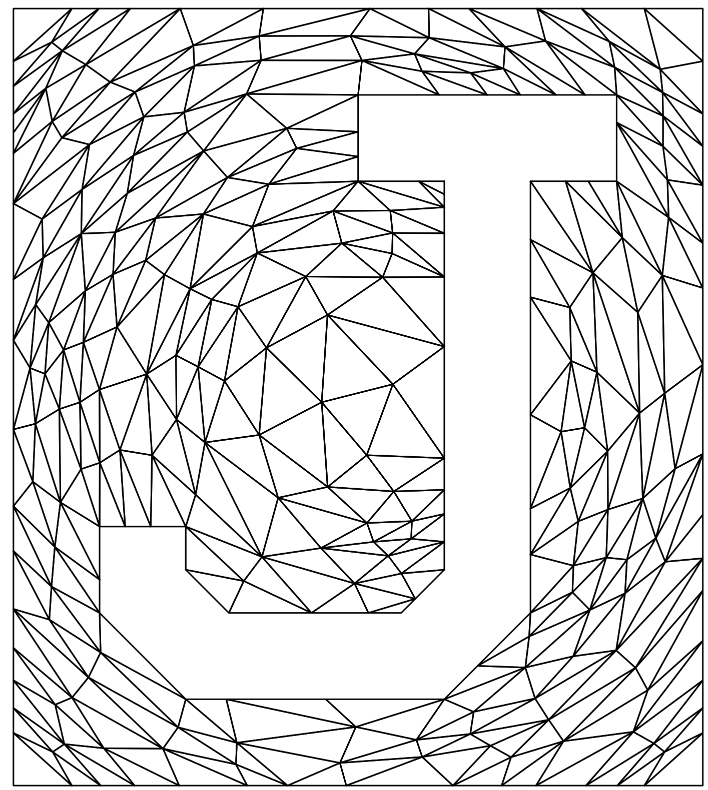




Main Result

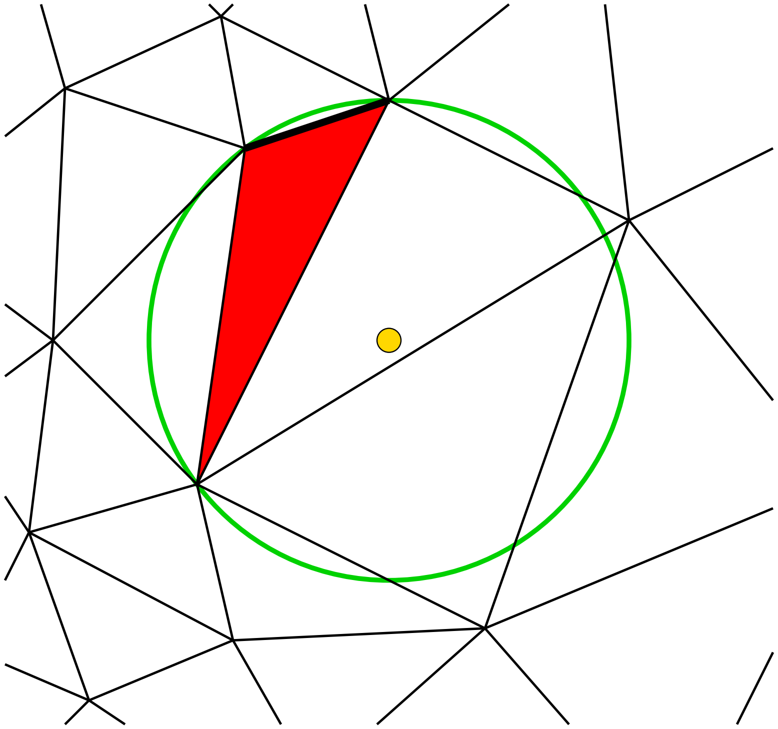
If metric tensor M is smooth with bounded derivatives, no triangle has angle < 20° as measured by any point in the triangle.





Why Does It Work?

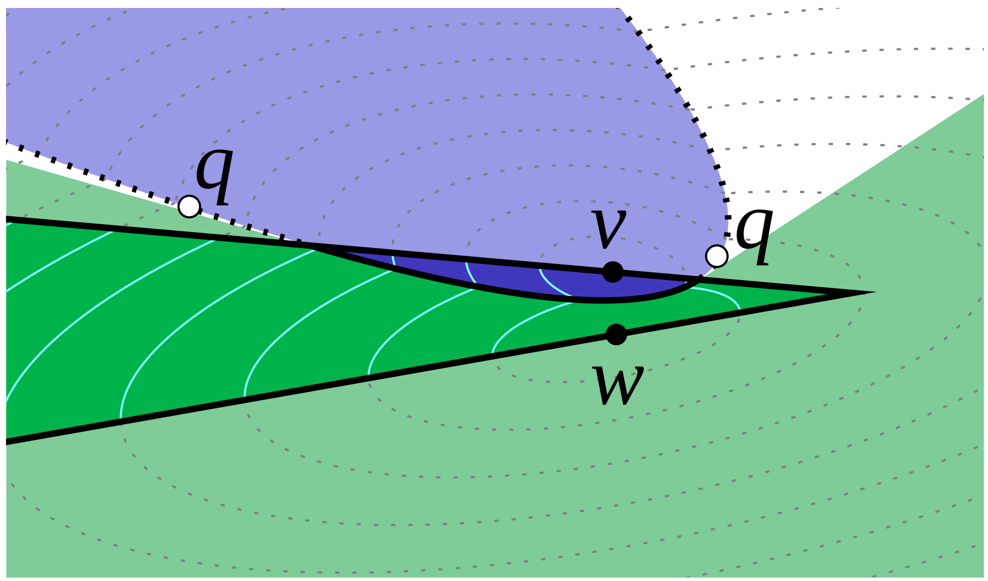
- It attacks every bad triangle and topological irregularity. Therefore, it will either succeed or refine forever.
- A bad triangle can exist only where a short edge lies beside a large gap. Filling the gap creates no shorter edges.



Why Does It Work?

If a point q on a Voronoi arc is not wedged, then either

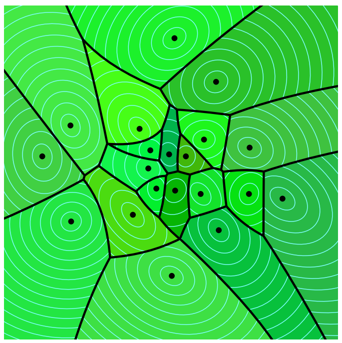
- q is far from v and w, or
- M_v and M_w are very different.



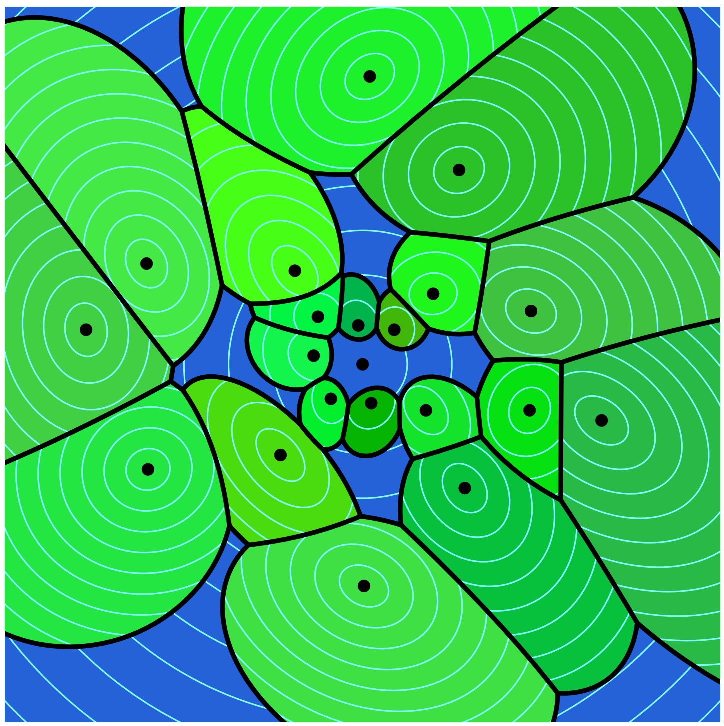
In the first condition, new edges are no shorter than the shortest existing edge.
Refinement will alleviate the second condition.

Loose Anisotropic Voronoi Diagrams

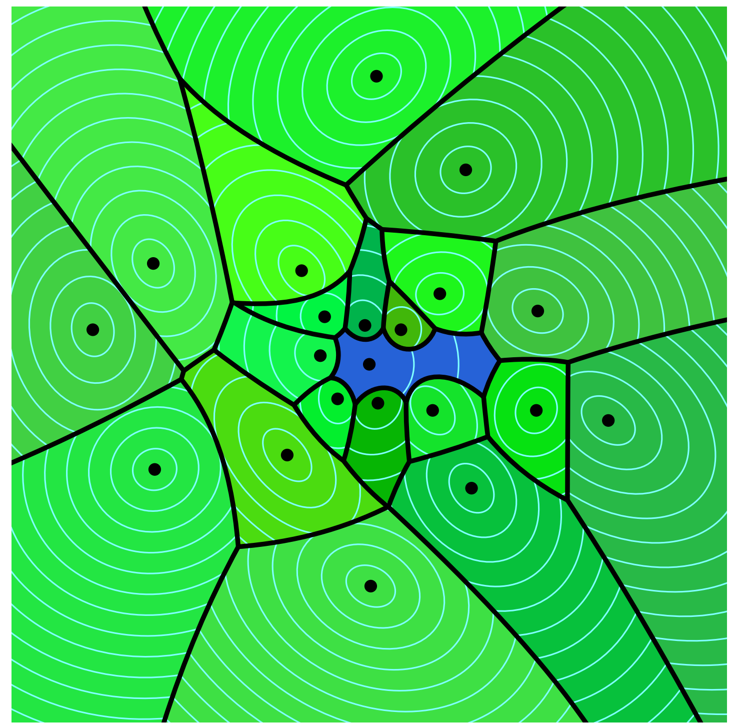
Fast local site insertion replaces $O(n^{2+\varepsilon})$ alg.



before



anisotropic
Voronoi diagram



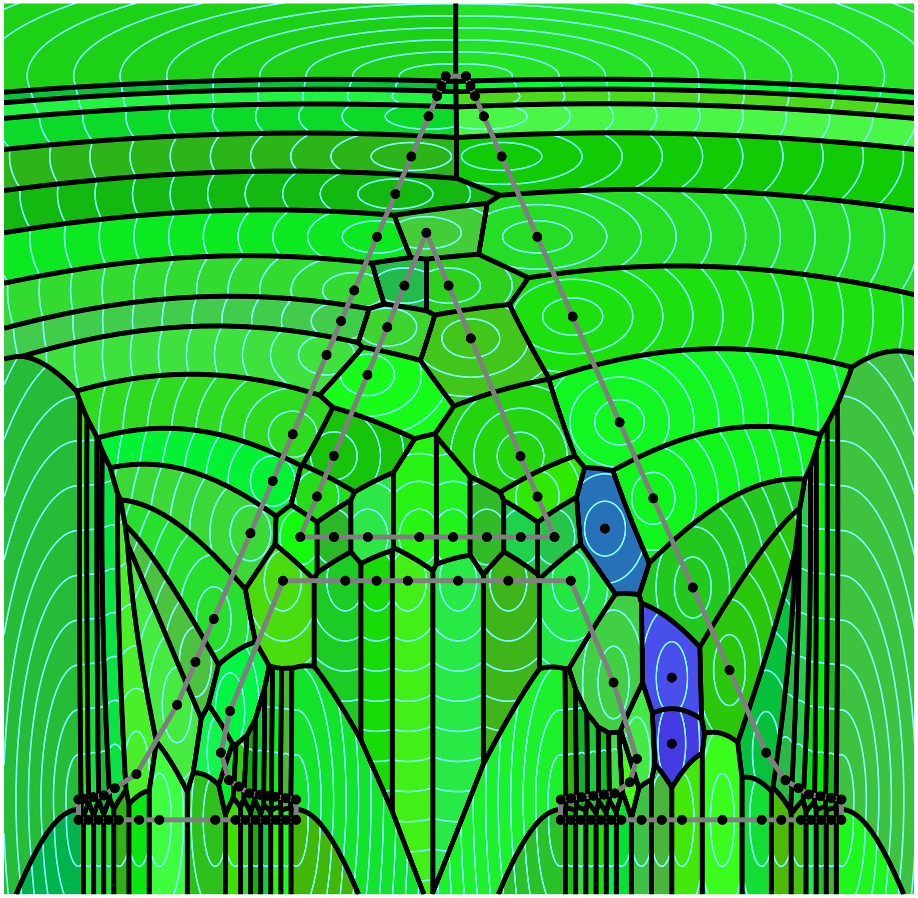
loose anisotropic
Voronoi diagram

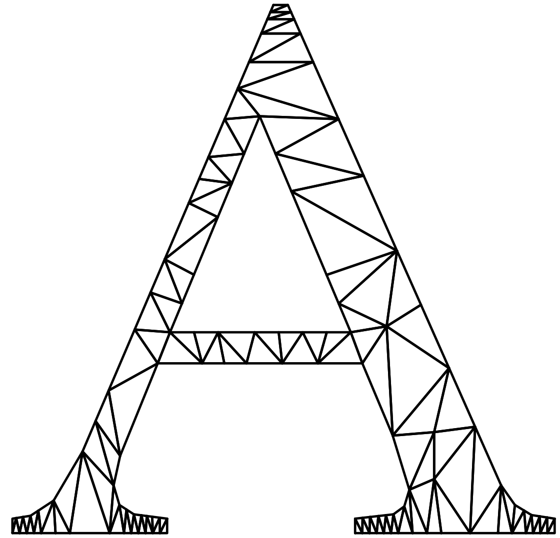
Conclusions

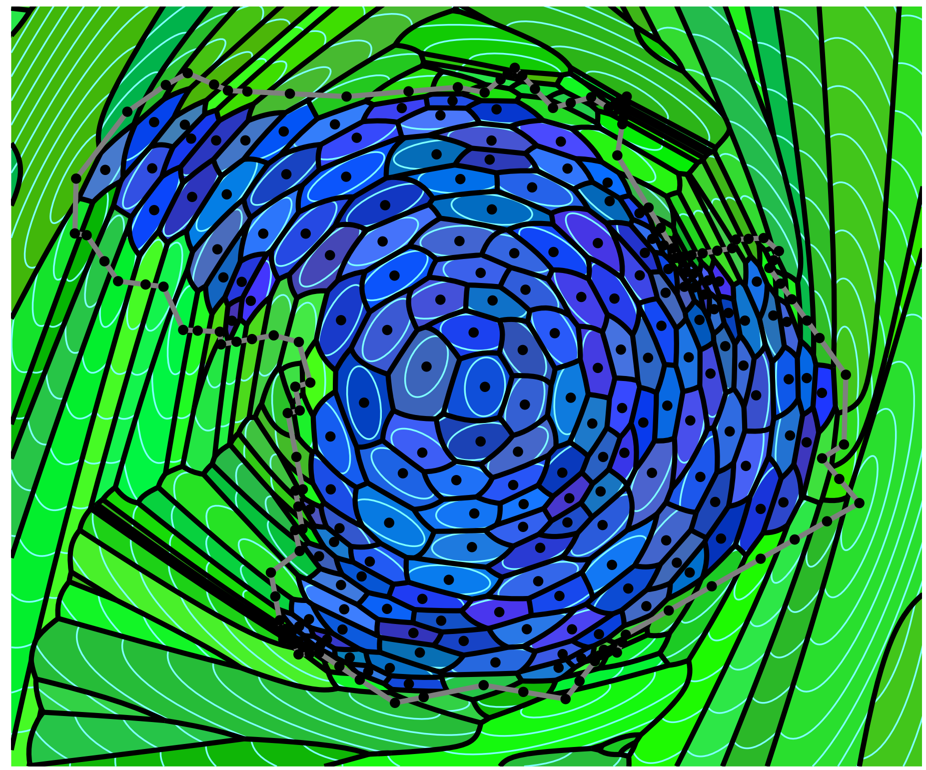
- Anisotropic Voronoi diagrams offer an elegant and fast way to define anisotropic ‘‘Delaunay’’ triangulations.
- The first theoretically guaranteed anisotropic mesh generation algorithm!

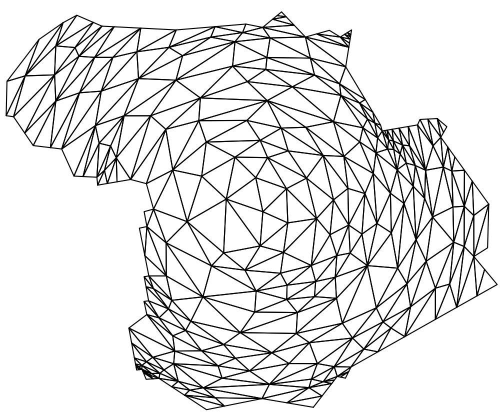
Future Work

- Should work in practice in 3D (though the theoretical properties don’t all follow).

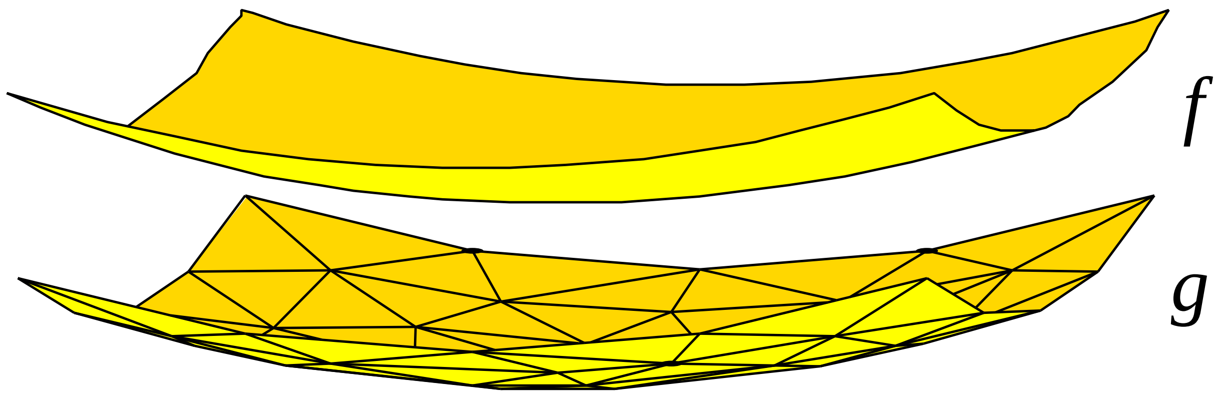








Anisotropy and Interpolation Error



H = Hessian of f.
Suppose $|\mathbf{d}^T H(p)\,\mathbf{d}| < \mathbf{d}^T C\,\mathbf{d}$ for any direction $\mathbf{d}$.
Let $E^2 = C$ with E symmetric positive definite.

You can judge the error $\| f - g \|_\infty$ of an element t by judging Et by isotropic error bounds/measures.