

# What is a Good Linear Finite Element?

Interpolation, Conditioning, Anisotropy, and Quality Measures

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# Two Communities: Mesh Generation and Error Analysis

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## Mesh generation people:

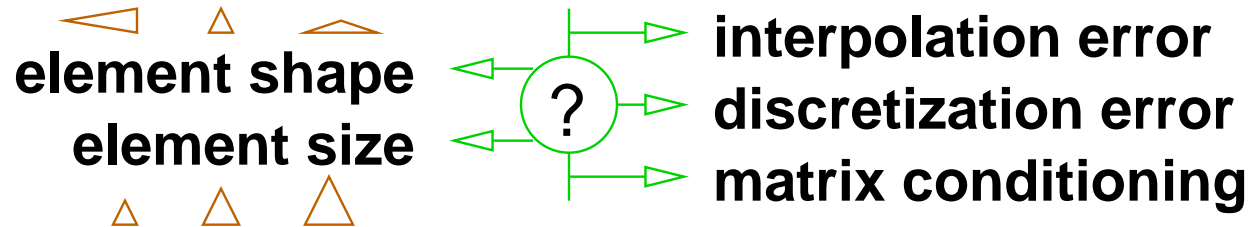
- Most don't *really* understand the goals of their own field!
  - Know from experience small & large angles are bad (and are faintly aware why).
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## Numerical analysts:

- Tend to derive *asymptotic* error bounds & estimators (functional analysis, embedding theorems) — *not very useful* to meshing people!
- Meshing people can't read functional analysis anyway.

# Error Bounds & Quality Measures

- The connections are still fuzzy.



(Especially in anisotropic cases.)

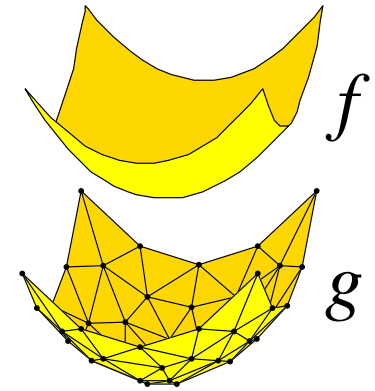
My goals:

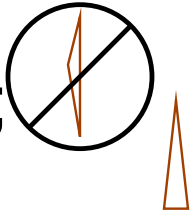
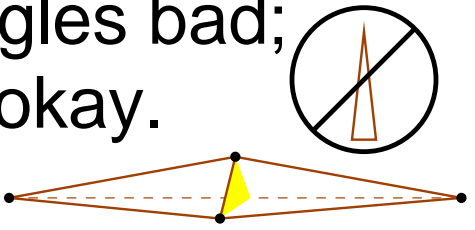
- (Nearly) tight bounds on worst–case errors, element stiffness matrix eigenvalues.
- Quality measures that can choose the better of two elements of intermediate quality.  
(Suitable for numerical optimization.)
- Guide mesh generators to make good elements.

# Three Criteria for Linear Elements

Let  $f$  be a function.

Let  $g$  be a piecewise linear interpolant of  $f$  over some triangulation.

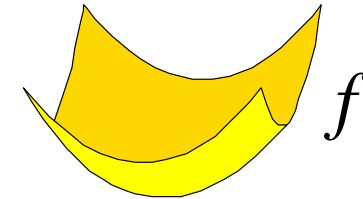


Criterion	
Interpolation error $\ f - g\ _\infty$	Size very important. Shape only marginally important.
Gradient interpolation error $\ \nabla f - \nabla g\ _\infty$	Size important. Large angles bad; small okay. 
Element stiffness matrix maximum eigenvalue $\lambda_{\max}$	Small angles bad; large okay. 

# Main Assumption

Curvature of  $f$  is bounded:

$$|f_{\mathbf{d}}''(p)| \leq c$$

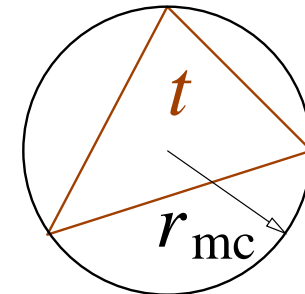
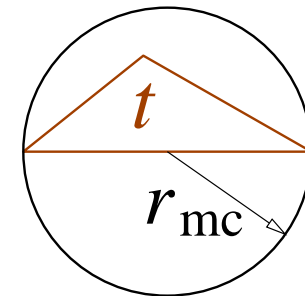


(second directional derivative along any direction  $\mathbf{d}$ )

Then over an element  $t$ ,

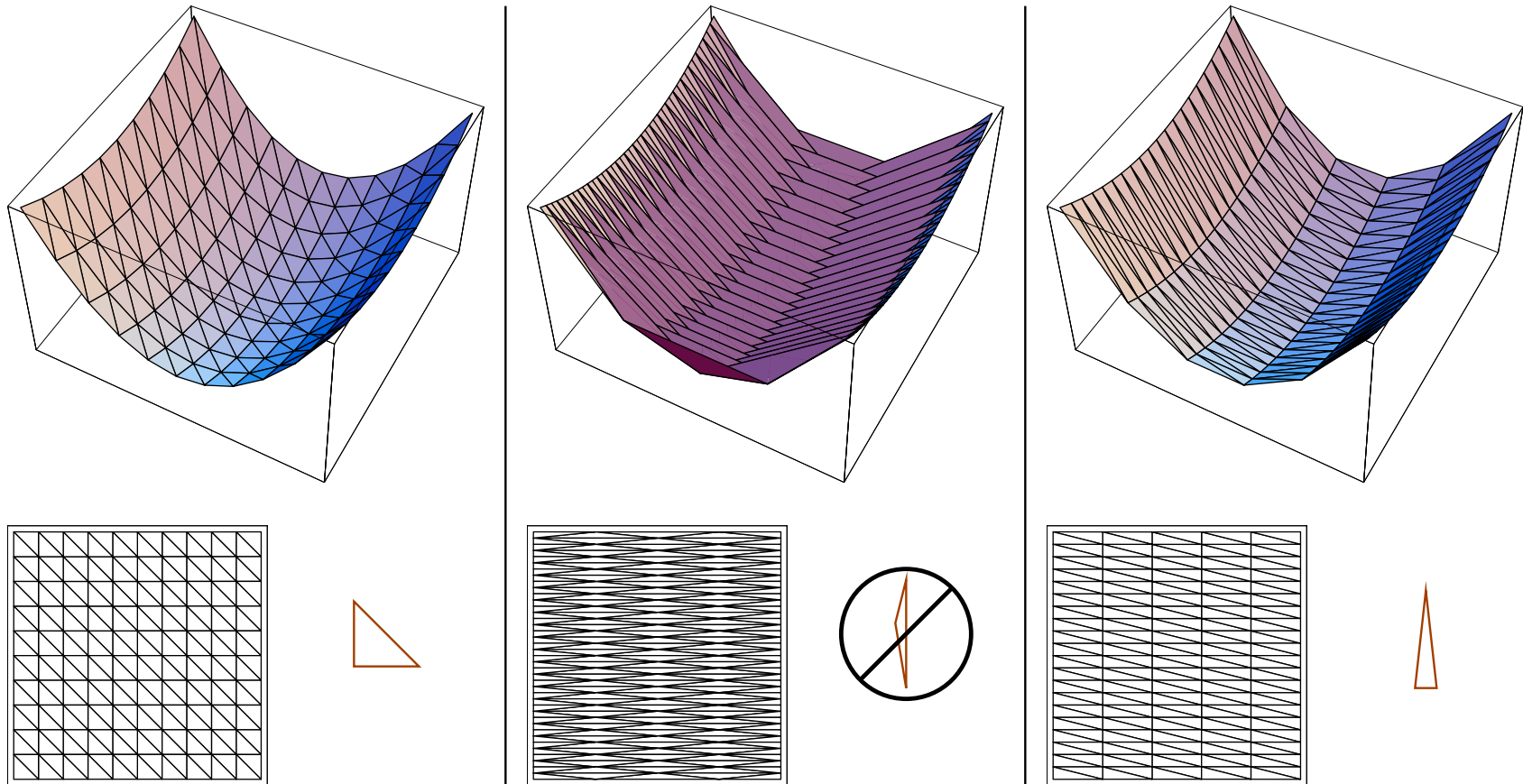
$$\|f - g\|_{\infty} \leq \frac{c}{2} r_{\text{mc}}^2$$

where  $r_{\text{mc}}$  is the radius of the min-containment circle/sphere of  $t$ . [Waldron 1998.]





Sharp for triangles, tetrahedra, higher dimensions...

# The Importance of Approximating Gradients Accurately



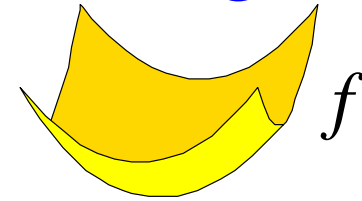
- $\|\nabla f - \nabla g\|_\infty$  affects discretization error in FEM.
- In mechanics,  $\nabla f$  is the strains.

# Classical Error Bounds on Gradients

- Approximation theory: error bound proportional to  $l_{\max} / r_{\text{in}}$ .
  - ← inradius of element 
  - ← maximum edge length of element
- Not asymptotically tight – overestimates error for elements with small angles. 
- Functional analysis: asymptotically tight error bound for triangles [Babuška and Aziz 1976]. And tetrahedra [Jamet 1976, Krížek 1992]?
  - But nobody knows the constant!

# Error Bound: Gradients on Triangles

(Same assumption: bounded curvature.)



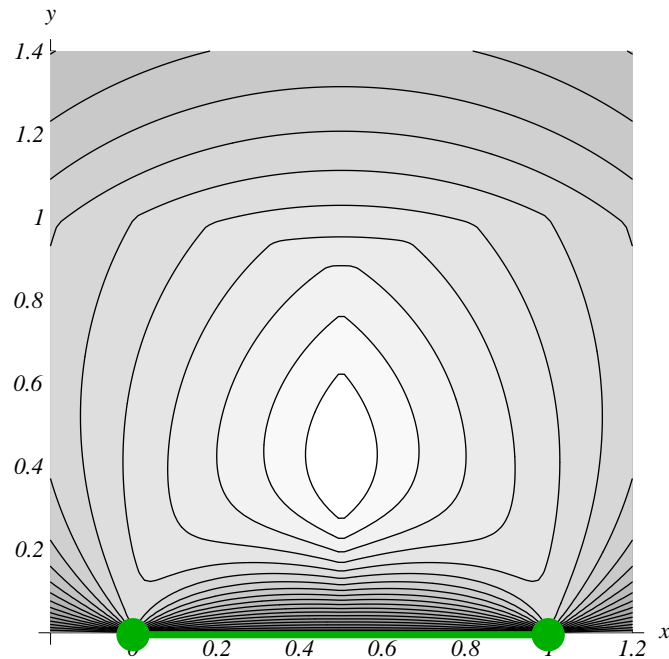
Over a triangle  $t$ ,

edge lengths      inradius of  $t$  

$$\|\nabla f - \nabla g\|_\infty \leq c \frac{l_{\max} l_{\text{med}} (l_{\min} + 4r_{\text{in}})}{4A}$$

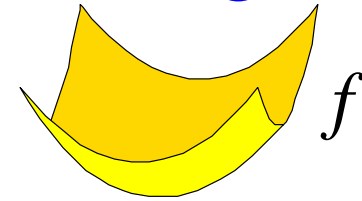
Area of  $t$  

- Bound tight within factor of 2.



# Error Bound: Gradients on Triangles

(Same assumption: bounded curvature.)

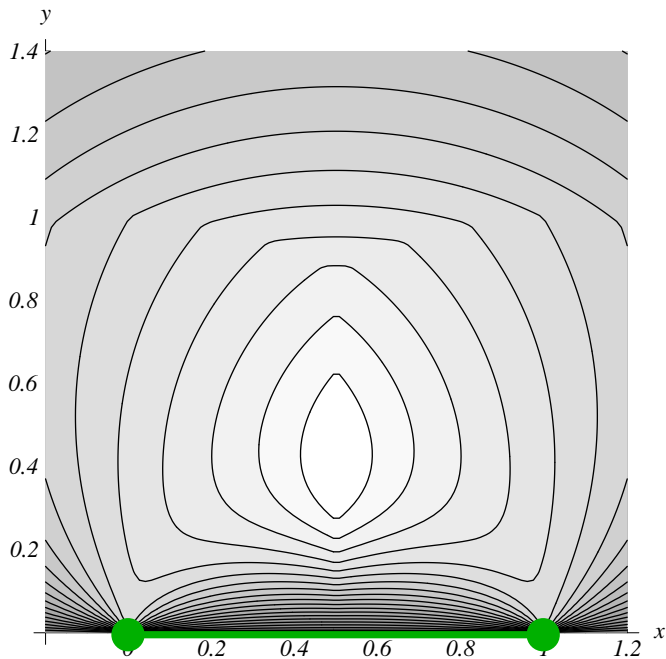


Over a triangle  $t$ ,

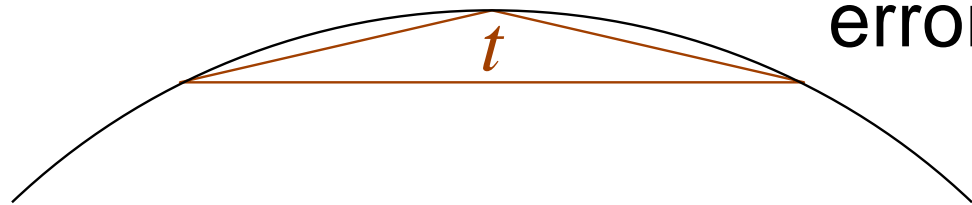
$$\|\nabla f - \nabla g\|_\infty \leq c \frac{l_{\max} l_{\text{med}} (l_{\min} + 4r_{\text{in}})}{4A} < 3c r_{\text{circ}}$$

edge lengths inradius of  $t$

Area of  $t$  Circumradius of  $t$



- Bound tight within factor of 2.
- Angle near  $180^\circ \rightarrow$   
large circumradius  $\rightarrow$  large error.



# Error Bound: Gradients on Tetrahedra

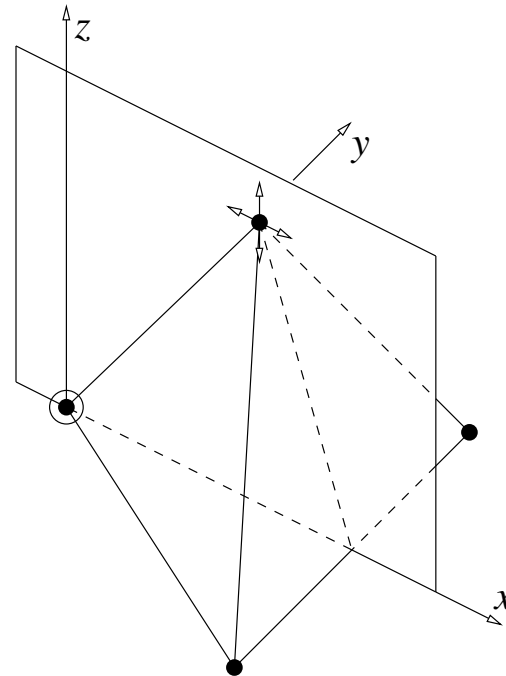
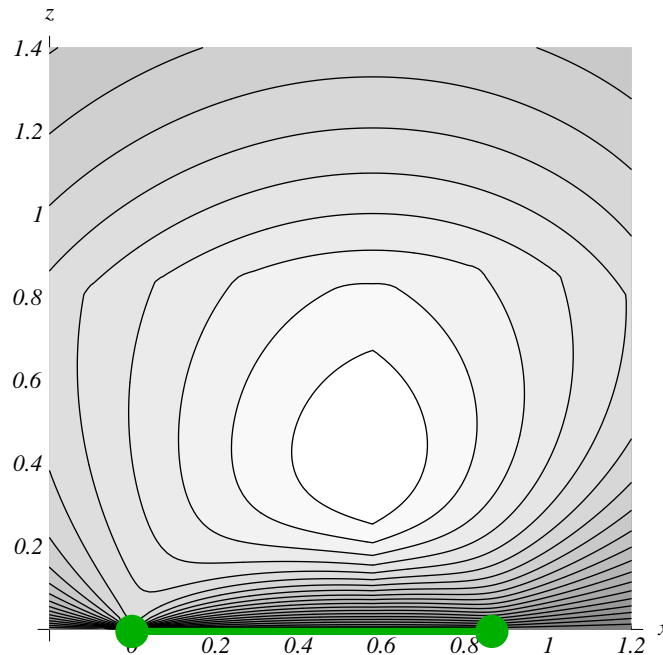
Over a tetrahedron  $t$ ,

Edge lengths of  $t$

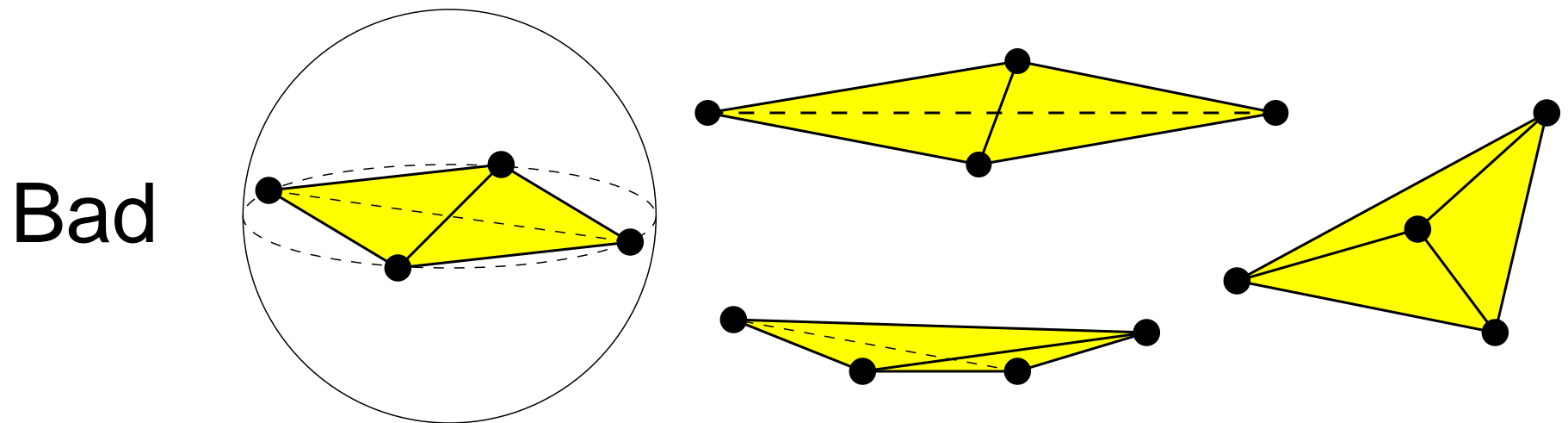
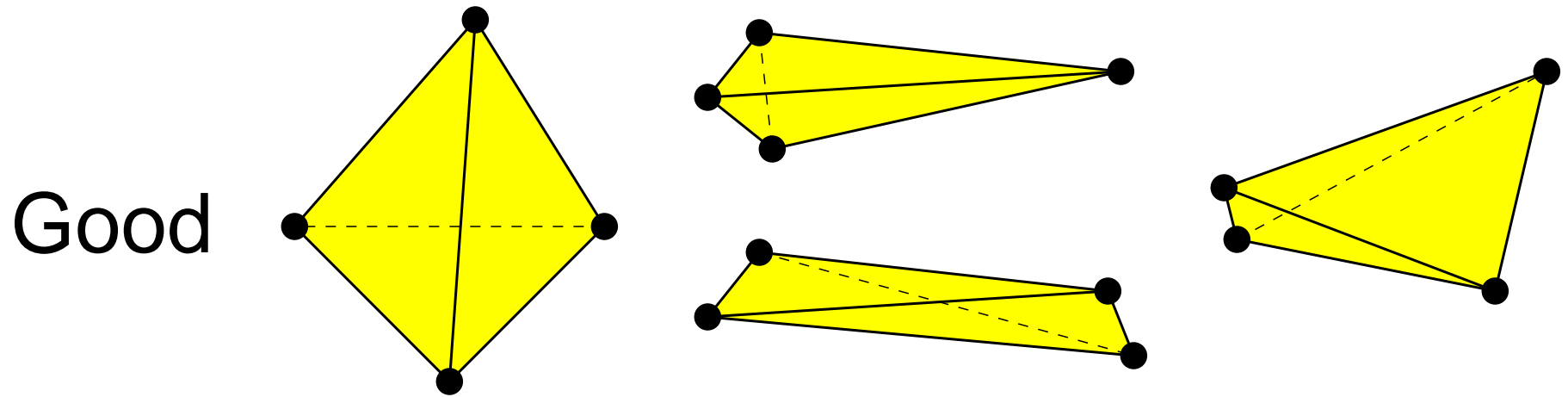
$$\|\nabla f - \nabla g\|_\infty \leq c \frac{\frac{1}{6V} \sum_{1 \leq i < j \leq 4} A_i A_j l_{ij} + \max_i \sum_{j \neq i} A_j l_{ij}}{\sum_{m=1}^4 A_m}$$

Volume of  $t$

Face areas of  $t$



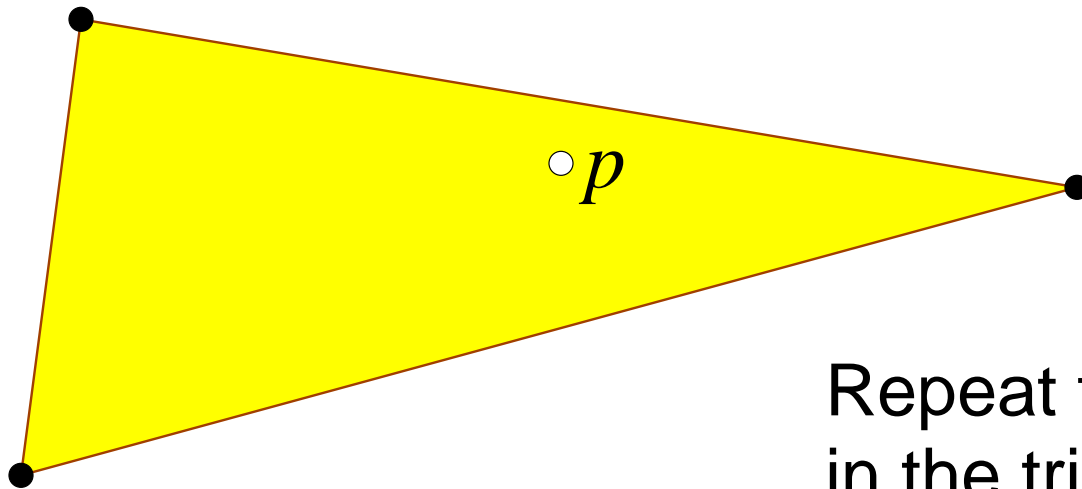
# Good and Bad Tetrahedra for Interpolation



# Deriving the New Gradient Error Bounds

Start with standard approximation theory:

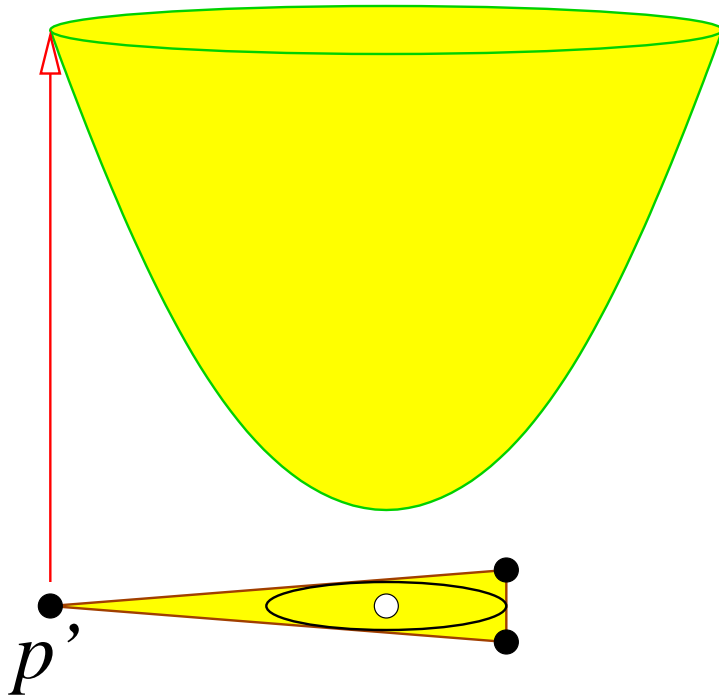
- Choose a point  $p$ .
- Take Taylor expansion of  $f-g$  about  $p$ .
- Set it to zero at element vertices ( $d+1$  equations).
- Eliminate  $f(p)-g(p)$  term from equations.
- Curvature bounds yield naive bound on  $\|\nabla f(p) - \nabla g(p)\|_\infty$ .



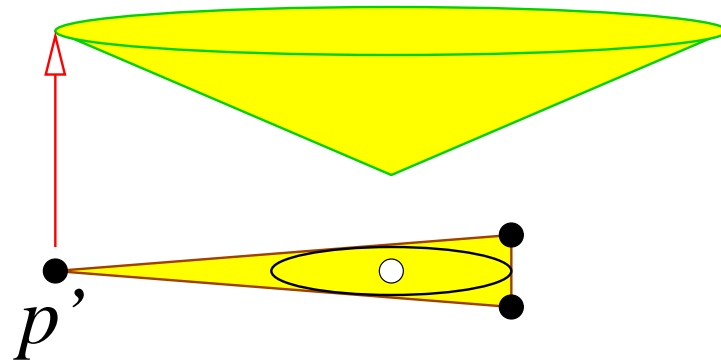
Repeat for each point  $p$   
in the triangle...

# Deriving the New Gradient Error Bounds

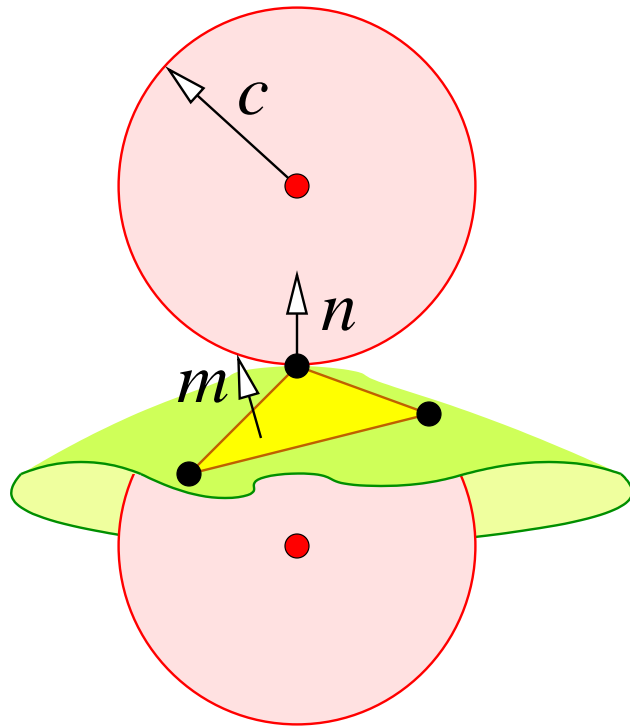
Naive bound on  $\|\nabla f(p) - \nabla g(p)\|_\infty$  is parabolic. Worst point  $p'$  gives standard  $l_{\max}/r_{\text{in}}$  bound.



But the naive error bound is minimized at the incenter. Curvature of  $f$  is bounded, so gradient of  $\|\nabla f - \nabla g\|_\infty$  is bounded.



# Error Bound: Triangle Normals on Surfaces



How much can triangle normal  $m$  deviate from surface normal  $n$ ?

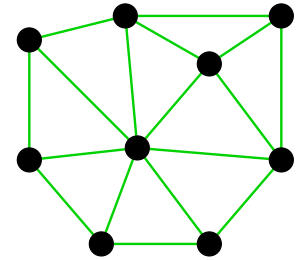
Assumption: spheres tangent to surface with radius  $c$  do not enclose any portion of surface.

Angle between  $m$  and  $n$  (at any vertex) is at most

$$\alpha + \arcsin\left(\frac{2}{\sqrt{3}} \sin 2\alpha\right) + \arcsin \frac{l_{\text{med}}}{2c}, \quad \alpha = \arcsin \frac{r_{\text{circ}}}{c}.$$

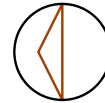
[Amenta, Choi, Dey, Leekha 2002.]

# Delaunay Optimality

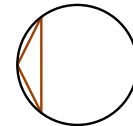


A set of vertices has many triangulations.

- In any dimensionality, the Delaunay triangulation minimizes the largest  $r_{mc}$ .

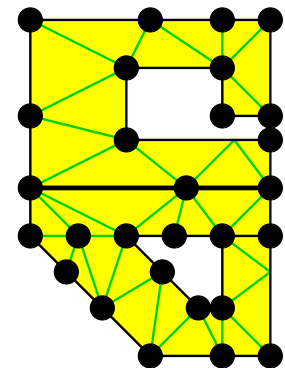


- In two dimensions, the Delaunay triangulation minimizes the largest  $r_{circ}$ .



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A domain has many triangulations that respect its boundaries. Among these, the constrained Delaunay triangulation is optimal.



# Conditioning of Global Stiffness Matrix

- $\lambda_{\max}$  :
- Dominated by the single worst element.
  - Depends on shape of element(s).
  - 2D: Independent of element size.
  - 3D: Largest element usually dominates.
- 

- $\lambda_{\min}$  :
- Relatively independent of shape.
  - Directly proportional to areas/volumes of elements.
  - Somewhere between smallest and largest elements (times a constant).

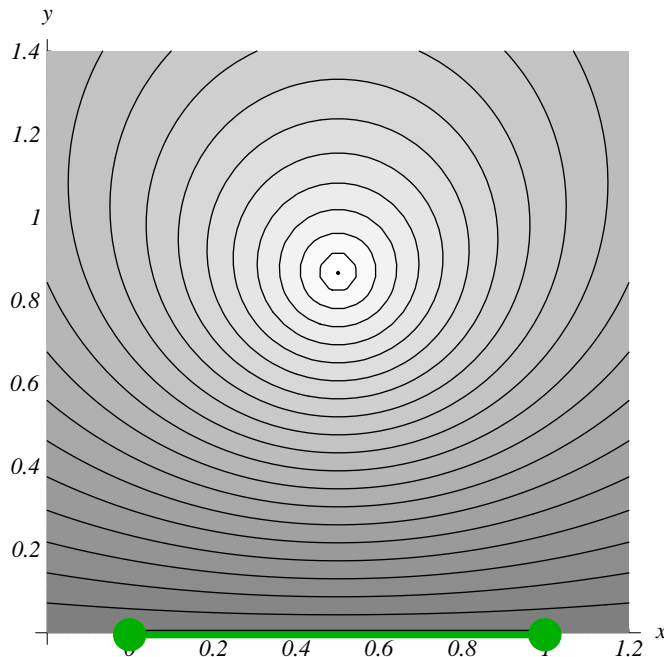
# Conditioning: Maximum Eigenvalue of Element Stiffness Matrix

(for Poisson's Equation)

edge lengths

$$\lambda_{\max} = \frac{l_1^2 + l_2^2 + l_3^2 + 2\sqrt{(l_1^2 + l_2^2 + l_3^2)^2 - 48A^2}}{8A}$$

Area of  $t$

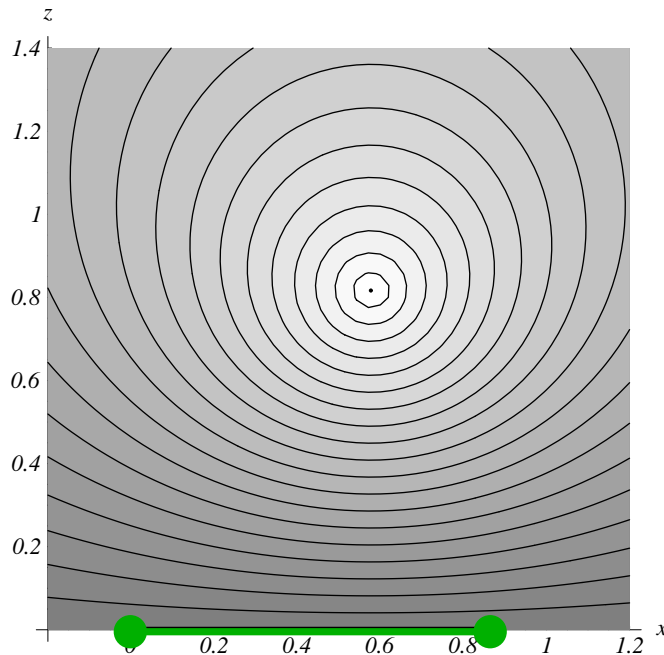


- Maximum eigenvalue is a quality measure that prefers equilateral triangles.
- Small angles are deleterious.

# Maximum Eigenvalue in 3D

(for Poisson's Equation)

- Eigenvalue for tetrahedron requires solving a cubic equation.
- Eigenvalue smallest for equilateral tetrahedra.

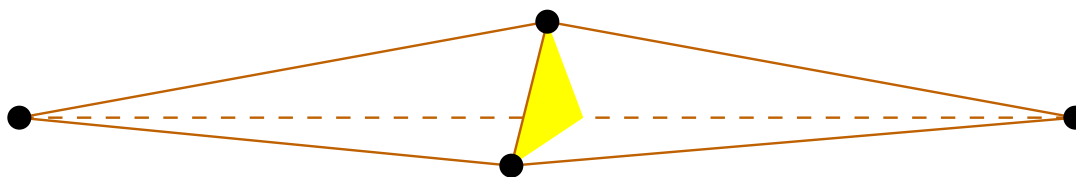


- Dihedral angles, not planar angles, are related to quality.
- It's a “well-known fact” that both small and large dihedral angles hurt conditioning...

# WRONG!!!

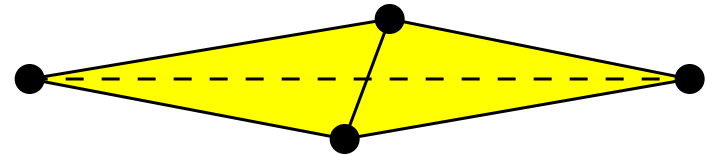
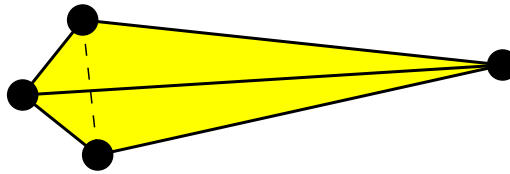
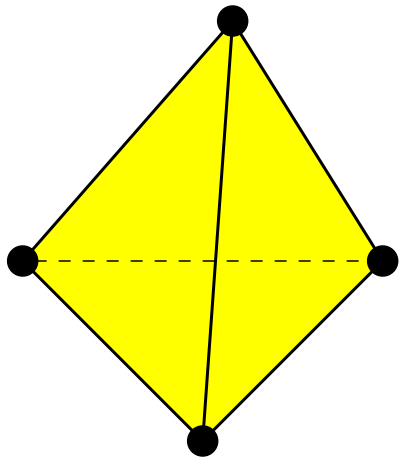
## Surprise #1

A tetrahedron can have a dihedral angle arbitrarily close to  $180^\circ$  with no dihedral smaller than  $60^\circ$ .  
Such a tetrahedron does not hurt conditioning at all!

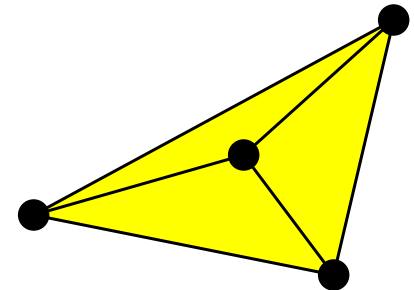
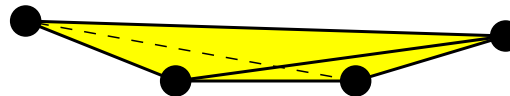
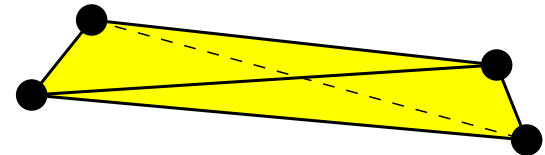
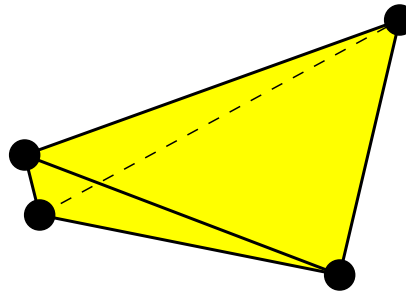
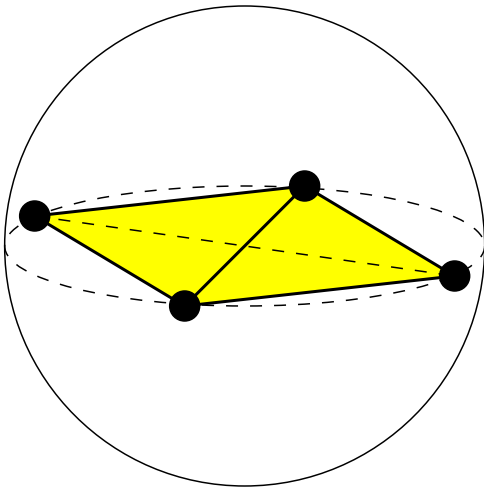


# Good and Bad Tetrahedra for Conditioning

Good

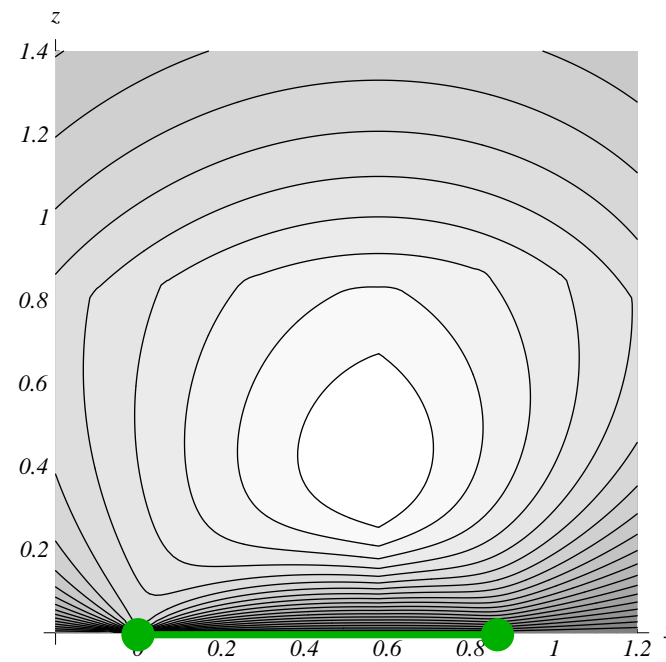
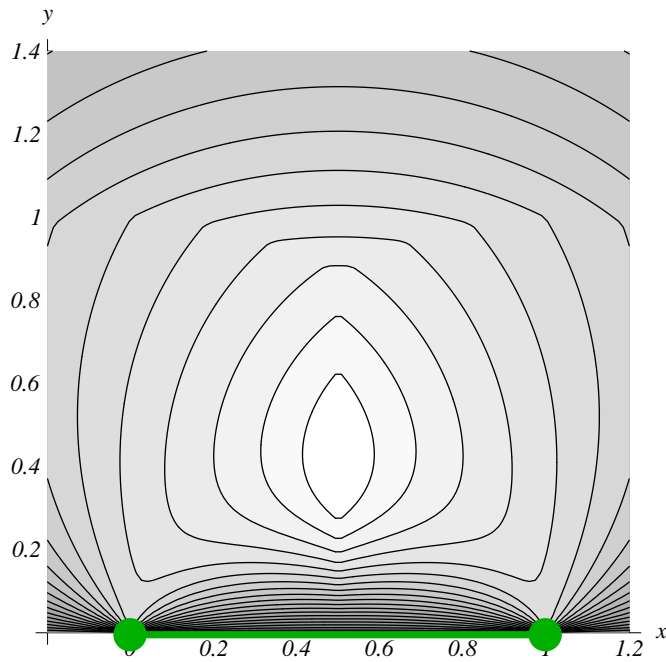


Bad



# Quality Measures

- Used to evaluate & choose elements.
- Reciprocal of interpolation error or max eigenvalue.
- Behave well as objective functions for mesh smoothing by numerical optimization.

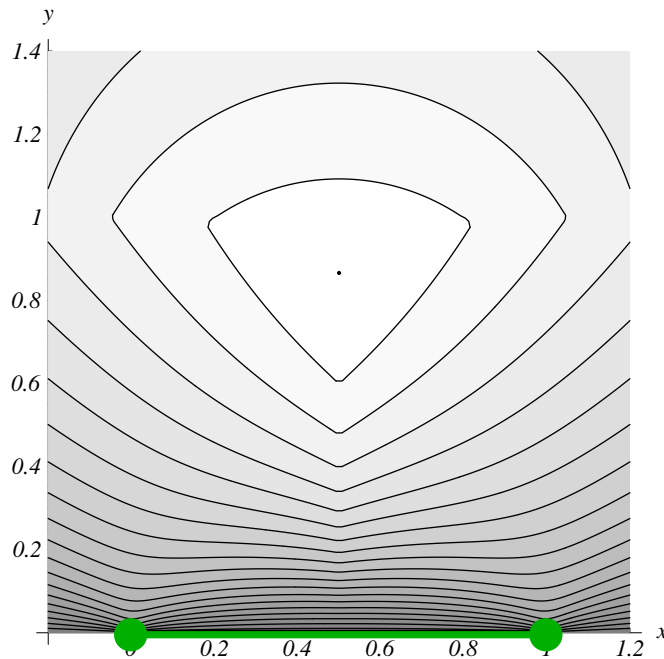


triangles  $\|\nabla f - \nabla g\|_\infty$  tetrahedra

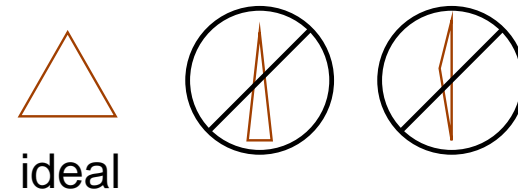
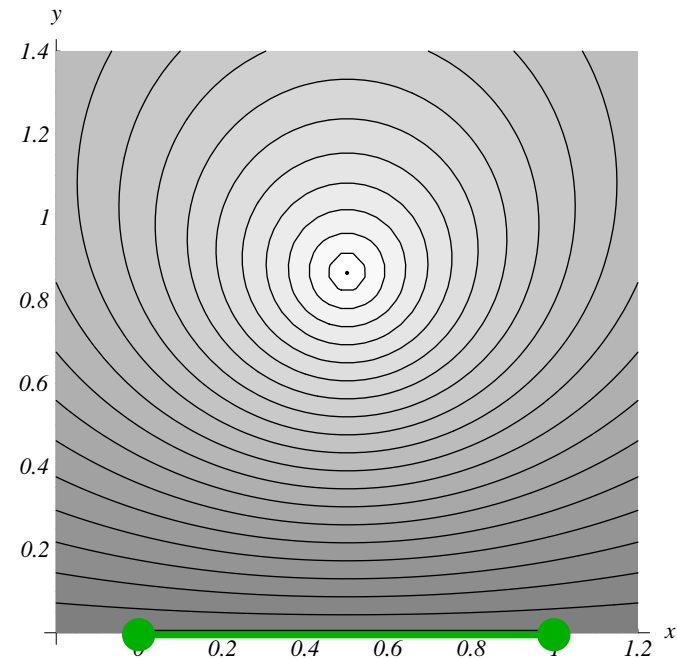
# Scale-Invariant Quality Measures

...measure how effective an element's shape is for a fixed amount of area.

$$\|\nabla f - \nabla g\|_\infty$$



$$\lambda_{\max}$$



# Two Types of Quality Measures

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## Scale-invariant quality measures:

- Separate the effects of shape from the effects of size.
  - Popular in mesh generation because they're easy to understand.
- 

## Size-and-shape measures & error bounds:

- Incorporate the effects of shape and size in one number.
- Require more understanding of application.
- What I advocate for most purposes.

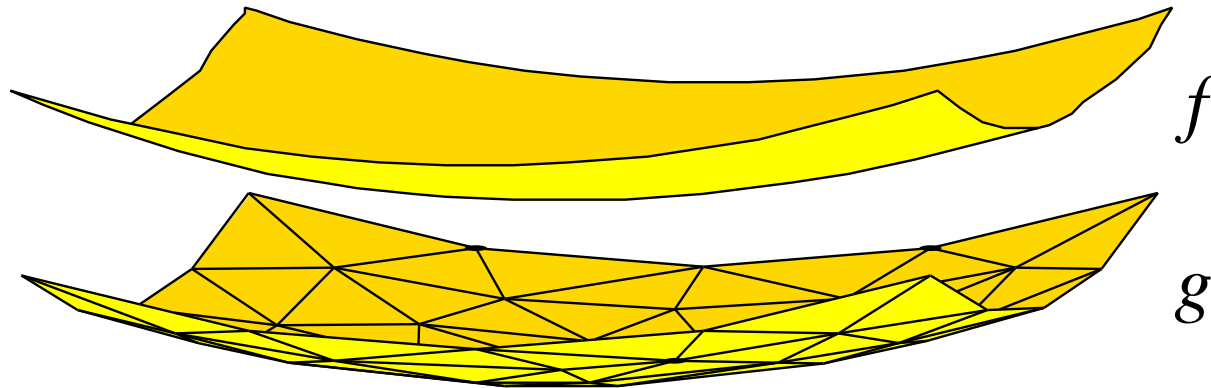
## Why Scale-Invariant Measures are Misleading

- Interpolation/discretization error and (in 3D) conditioning depend on element size too!
- It's okay (usually) for smaller elements to be worse shaped than big ones. Smoothing, cleanup, Delaunay refinement should take this into account.
- Example: 2D scale-invariant measure for interpolated gradients suggests “minimize large angles.” But size-and-shape measure says “minimize circumradii” → Delaunay.

# Uses of Error Bounds & Quality Measures

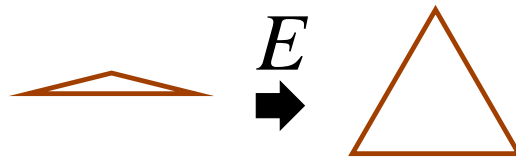
- **Mesh refinement:** Refine element if either  $\|f - g\|_\infty$  or  $\|\nabla f - \nabla g\|_\infty$  is too large. Use Delaunay refinement if conditioning/shape bad.
- **Mesh smoothing:** Quality measures are designed for numerical optimization of vertex positions. Smoother, slightly weaker measures available.
- **Topological mesh improvement:** Use quality measures plus refinement bound to judge elements.
- **Vertex placement in advancing front meshing.**

# Anisotropy and Interpolation Error

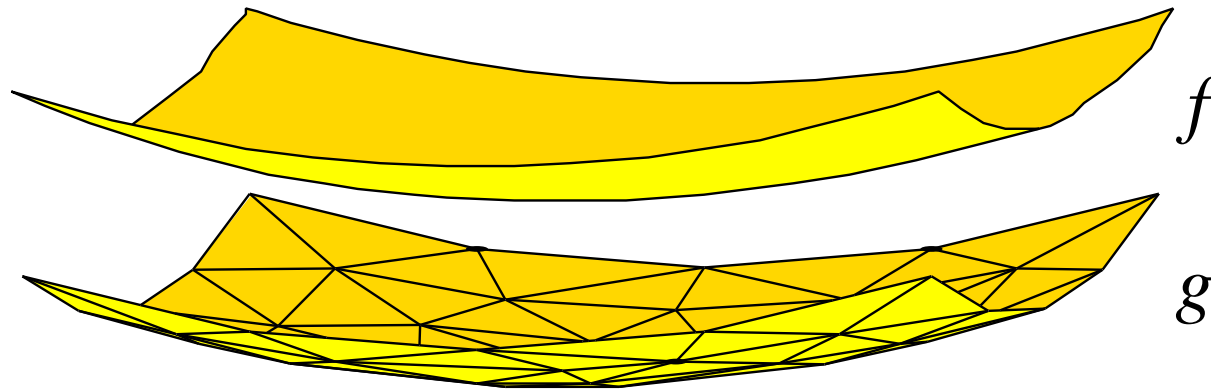



$H = \text{Hessian of } f$ . Let  $E^2 = H$  with  $E$  symmetric pos-def.

You can judge the error  $\|f - g\|_\infty$  of an element  $t$  by judging  $Et$  by isotropic error bounds/measures.

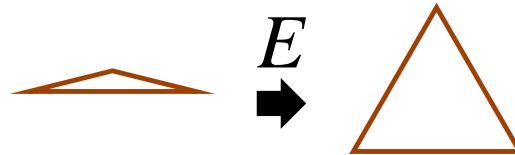


# Anisotropy and Gradient Interpolation



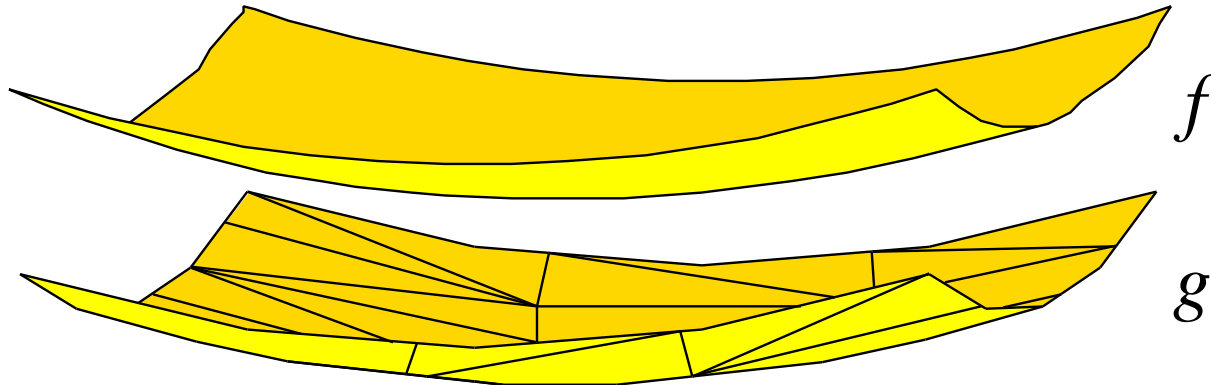
Large angles are fine if aligned correctly and not too large. 

A good element  $t$  for controlling  $\|\nabla f - \nabla g\|_\infty$  is one for which  $Et$  has no large angle.



But...

## Surprise #2: Superaccurate Gradients

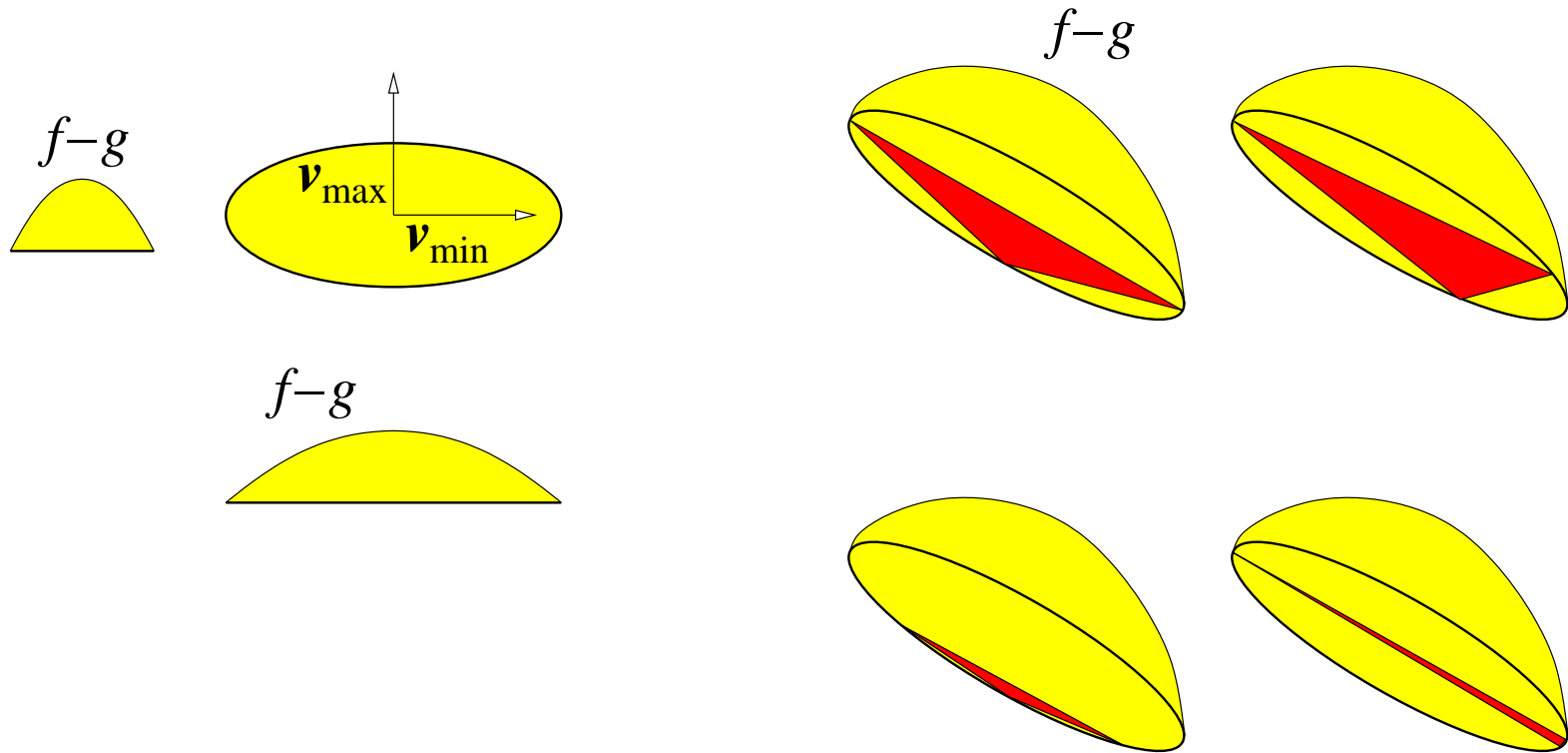


Longer, thinner elements than expected sometimes give the best accuracy! (For a fixed # of elements.)



But no large angles allowed for these extra-long ones. And they must be very precisely aligned.

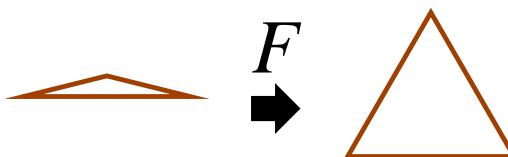
# Superaccurate Gradients




# Anisotropy and Conditioning

Ideal element for stiffness matrix depends on anisotropy of the PDE, not the solution.

$$-\nabla \cdot B \nabla f = 0$$

$$F^2 = B^{-1}$$


Large and small angles are fine if  $Ft$  looks good by isotropic standards. 

Equilateral elements can be quite bad for conditioning.

## Surprise #3: Anisotropy Blues

- Interpolation and conditioning don't always agree on the ideal aspect ratio or orientation!
- Superaccuracy implies that discretization error sometimes disagrees with both interpolation and conditioning!

Advice:

- Interpolation/discretization error can always be improved by refining. Conditioning cannot.
- With finite elements, always choose small discretization error over interpolation.

## Concluding Questions

- What about quadratic triangles and tetrahedra?
- What about bilinear quads and trilinear hexes?
- What if the curvature bound is in the  $L_2$  norm?