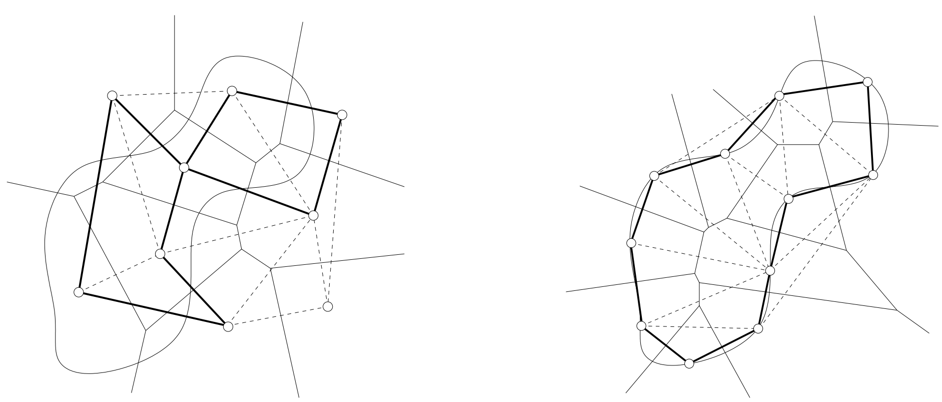
Chapter 13

Restricted Delaunay triangulations of surface samples

The restricted Delaunay triangulation is a subcomplex of the three-dimensional Delaunay triangulation that has proven itself as a mathematically powerful tool for surface meshing and surface reconstruction. Consider a smooth surface $\Sigma \subset \mathbb{R}^3$ and a finite sample S on Σ. If S is dense enough, as dictated by the local feature size over Σ, then the restricted Delaunay triangulation is a triangulation of Σ by Definition 12.6—it has an underlying space that is guaranteed to be topologically equivalent to Σ. Moreover, the triangulation lies close to Σ and approximates it geometrically. The theory of restricted Delaunay triangulations of surface samples lays the foundation for the design and analysis of Delaunay refinement mesh generation algorithms in the subsequent chapters, which use incremental vertex insertion not only to guarantee good element quality, but also to guarantee that the mesh is topologically correct and geometrically close to the surface, with accurate surface normals.



(a) (b)

Figure 13.1: Restricted Delaunay triangulations, indicated by bold edges. These edges are included because the curve intersects their Voronoi dual edges. Dashed edges are Delaunay but not restricted Delaunay. The restricted Delaunay triangulation at right is a passable reconstruction of the curve.

13.1 Restricted Voronoi diagrams and Delaunay triangulations

Restricted Delaunay triangulations are defined in terms of their duals, the restricted Voronoi diagrams; we must understand the latter to understand the former. The *restriction* $Q|_{\mathbb{T}}$ of a point set $Q \subset \mathbb{R}^d$ to a topological subspace $\mathbb{T} \subset \mathbb{R}^d$ is simply $Q \cap \mathbb{T}$. The *restriction* $\mathcal{C}|_{\mathbb{T}}$ of a complex $\mathcal{C}$ to a topological subspace $\mathbb{T}$ is $\{g \cap \mathbb{T} : g \in \mathcal{C}\}$, the complex found by taking the restriction of each cell in $\mathcal{C}$. The restricted Voronoi diagram is defined accordingly.

Definition 13.1 (restricted Voronoi diagram)**.** Let $S \subseteq \mathbb{R}^d$ be a finite set of sites, and let $\mathbb{T} \subseteq \mathbb{R}^d$ be a topological subspace of a Euclidean space. The *restricted Voronoi cell* $V_u|_{\mathbb{T}}$ of a site $u \in S$ is $V_u \cap \mathbb{T} = \{p \in \mathbb{T} : \forall w \in S, d(u, p) \leq d(w, p)\}$, the restriction of u's Euclidean Voronoi cell to $\mathbb{T}$. A *restricted Voronoi face* $V_{u_1 \ldots u_j}|_{\mathbb{T}}$ is a nonempty restriction of a Voronoi face $V_{u_1 \ldots u_j} \in \operatorname{Vor} S$ to $\mathbb{T}$—that is, $V_{u_1 \ldots u_j}|_{\mathbb{T}} = V_{u_1 \ldots u_j} \cap \mathbb{T}$. The *restricted Voronoi diagram* of S with respect to $\mathbb{T}$, denoted $\operatorname{Vor}|_{\mathbb{T}} S$, is the cell complex containing every restricted Voronoi face.

Restricted Delaunay triangulations are defined *not* by restricting Delaunay simplices to a topological subspace, but by dualizing the restricted Voronoi diagram.

Definition 13.2 (restricted Delaunay)**.** A simplex is *restricted Delaunay* if its vertices are in S and together they generate a nonempty restricted Voronoi face. In other words, $\operatorname{conv}\{u_1, \ldots, u_j\}$ is restricted Delaunay if $V_{u_1 \ldots u_j}|_{\mathbb{T}}$ is nonempty. If S is generic, the *restricted Delaunay triangulation* of S with respect to $\mathbb{T}$, denoted $\operatorname{Del}|_{\mathbb{T}} S$, is the simplicial complex containing every restricted Delaunay simplex. If S is not generic, a restricted Delaunay triangulation of S is a simplicial complex containing every restricted Delaunay simplex in some particular $\operatorname{Del} S$.

Figure 13.1(a) shows a restricted Delaunay triangulation with respect to a loop in the plane. Typically, no Voronoi vertex lies on the curve, so there are no restricted Delaunay triangles. Figure 13.1(b) shows a different restricted Delaunay triangulation with respect to the same loop. The vertices in the latter example are a fairly good sample of the curve, and the restricted triangulation is also a loop. In general, our goal is to sample a space so that $|\operatorname{Del}|_{\mathbb{T}} S|$ is homeomorphic to $\mathbb{T}$, as it is here. More powerfully, there is a homeomorphism that maps every sample point to itself.

It is not obvious that the set of all restricted Delaunay simplices is really a complex. Observe that if $\operatorname{Del}|_{\mathbb{T}} S$ contains a simplex σ, it contains every face of σ because every subset of σ's vertices generates a Voronoi face that includes the face dual to σ and, therefore, intersects Σ. If S is generic, then $\operatorname{Del}|_{\mathbb{T}} S \subseteq \operatorname{Del} S$, because every face of $\operatorname{Del}|_{\mathbb{T}} S$ dualizes to a face of $\operatorname{Vor}|_{\mathbb{T}} S$, which is induced by restricting a face of $\operatorname{Vor} S$, which dualizes to a face of $\operatorname{Del} S$. Therefore, $\operatorname{Del}|_{\mathbb{T}} S$ is a simplicial complex, and a standard Delaunay triangulation algorithm is a useful first step in constructing it.

If S has multiple Delaunay triangulations, the set of all restricted Delaunay simplices might not form a complex, so we compute a particular $\operatorname{Del} S$ and choose its restricted Delaunay simplices to form $\operatorname{Del}|_{\mathbb{T}} S$. Alternatively, we could use a subcomplex of the Delaunay subdivision.

In this chapter, the topological space $\mathbb{T}$ is usually a smooth surface Σ embedded in three dimensions. (In later chapters $\mathbb{T}$ will sometimes be a volume or a piecewise smooth complex.) In most applications of restricted Delaunay triangulations, the site set S is a sample of Σ, but our definitions do not require it; a set of sites $S \not\subset \Sigma$ is sometimes useful to model noisy point samples or to remesh a polyhedral surface that approximates a smooth surface. In this book's applications, however, we will usually generate S on Σ.

We say that a Voronoi k-face intersects Σ *generically* or *transversally* if at each point of the intersection, the plane tangent to Σ does not include the affine hull of the Voronoi face. Such an intersection is a $(k-1)$-manifold with or without boundary. For example, a Voronoi edge may intersect Σ at one or more distinct points, and a Voronoi polygon may intersect Σ in one or more curves or loops. By contrast, intersections can be *degenerate* or *non-transverse*, such as a Voronoi polygon whose intersection with Σ is a single point or a 2-ball or a figure-8 curve, or a Voronoi vertex that intersects Σ. If a k-flat Π has a non-transverse intersection with a C^2-smooth surface Σ, there is a point in $\Pi \cap \Sigma$ at which Π is tangent to Σ. Non-transverse intersections can be eliminated by perturbing the surface; for example, if a Voronoi vertex lies on Σ, simply pretend it lies infinitesimally inside the surface.

Even though $\text{Del}|_\Sigma S$ is always a simplicial complex, it is not guaranteed to be a triangulation of Σ, or coherent in any way, unless we impose strong constraints on S. Ideally, each restricted Voronoi cell $V_p|_\Sigma$ would be a simple region homeomorphic to a disk. But if the sample S is not dense enough, a Voronoi cell V_p can reach through space to touch other portions of the surface, so $V_p|_\Sigma$ can have multiple connected components. Odder problems can occur; for example, imagine a surface in the shape of a sausage with just three sample points, one in the middle and one at each end. The restricted Voronoi cell of the sample point in the middle is topologically equivalent to an annulus instead of a disk.

This chapter studies how the topologies of $\text{Vor}|_\Sigma S$ and $\text{Del}|_\Sigma S$ are determined by the ways in which the faces of $\text{Vor}\, S$ intersect Σ, and how a dense sample S tames those intersections. Specifically, we require S to be a 0.08-sample of Σ (recall Definition 12.16)—though we hope the constant can be improved, and we observe that the algorithms are more forgiving in practice.

We begin with a well-known theorem in computational topology, the Topological Ball Theorem, which states that if every face of $\text{Vor}\, S$ intersects Σ nicely enough, then the underlying space of $\text{Del}|_\Sigma S$ is homeomorphic to Σ. Thus by Definition 12.6, $\text{Del}|_\Sigma S$ is a triangulation of Σ. This establishes the goal of most of the rest of the chapter: to establish how finely we must sample Σ to guarantee that the antecedents of the theorem hold.

Next, we prove a local result about restricted Voronoi vertices, which are the points where Voronoi edges pass through the surface. If every vertex of a restricted Voronoi cell $V_p|_\Sigma$ is close to its generating site $p \in S$, and every connected component of $V_p|_\Sigma$ has a vertex, then $V_p|_\Sigma$ and its restricted Voronoi faces are topological closed balls of appropriate dimensions. Specifically, if g is a k-face of V_p, the restricted Voronoi face $g|_\Sigma$ is a topological $(k-1)$-ball. A Voronoi edge intersects Σ at a single point; a Voronoi polygon intersects Σ in a single curve that is not a loop; and a Voronoi 2-cell intersects Σ in a topological disk.

The local result says nothing about a restricted Voronoi face that does not adjoin a restricted Voronoi vertex—for example, a circular face where a Voronoi polygon cuts a fingertip off Σ. We extend the local result to a global result. If S is dense enough—for example, if it is a 0.08-sample of Σ—and at least one restricted Voronoi vertex exists, then every Voronoi k-face either intersects Σ in a topological $(k-1)$-ball or does not intersect Σ

at all.

Having established that every face of $\operatorname{Vor} S$ intersects Σ nicely, we apply the Topological Ball Theorem to establish the homeomorphism of $|\operatorname{Del}|_\Sigma S|$ and Σ. We strengthen this result with another correspondence that is both topological and geometric. We construct an explicit isotopy relating $|\operatorname{Del}|_\Sigma S|$ and Σ, and show that each point in $|\operatorname{Del}|_\Sigma S|$ is displaced only slightly by this isotopy. Hence, $\operatorname{Del}|_\Sigma S$ is a geometrically good triangulation of Σ.

Finally, we prove additional results about the geometric quality of $\operatorname{Del}|_\Sigma S$: a bound on the circumradii of its triangles, a small deviation between the normal of a triangle and the normals to the surface at its vertices, and a large lower bound (near π) on the dihedral angles between adjoining triangles. These properties of $\operatorname{Del}|_\Sigma S$ are summarized in the Surface Discretization Theorem (Theorem 13.22) at the end of the chapter.

For the rest of this chapter, Σ is a *smooth surface*, our shorthand for a compact C^2-smooth 2-manifold without boundary, embedded in $\mathbb{R}^3$. Moreover, we assume that Σ is connected. A unit normal vector, written $\mathbf{n}_p$, is normal to Σ at a point p. All sets of restricted Delaunay vertices are samples $S \subset \Sigma$. Because we are exclusively concerned with three-dimensional space, *facet* will always mean 2-face, and a Voronoi facet is a Voronoi polygon. We use the following notation for many results and their proofs.

$$\begin{aligned}
\alpha(\varepsilon) &= \frac{\varepsilon}{1-\varepsilon}, \\
\beta(\varepsilon) &= \alpha(2\varepsilon) + \arcsin\frac{\varepsilon}{1-2\varepsilon} + \arcsin\left(\frac{2}{\sqrt{3}}\sin\left(2\arcsin\frac{\varepsilon}{1-2\varepsilon}\right)\right).
\end{aligned}$$

The first expression arises from both the Feature Translation Lemma (Lemma 12.2) and the Normal Variation Theorem (Theorem 12.8), and the second in part from the Triangle Normal Lemma (Lemma 12.14). We use them partly for brevity and partly in the hope they can be improved. Several proofs in this chapter are obtained by contradicting the following two conditions.

$$\begin{aligned}
\text{Condition A.1}: \quad \alpha(\varepsilon) &< \cos(\alpha(\varepsilon) + \beta(\varepsilon)) \\
\text{Condition A.2}: \quad \alpha(\varepsilon) &< \cos(\alpha(\varepsilon) + \arcsin\varepsilon + 2\beta(\varepsilon))
\end{aligned}$$

Condition A.1 is satisfied for $\varepsilon \le 0.15$, and Condition A.2 is satisfied for $\varepsilon \le 0.09$.

13.2 The Topological Ball Theorem

The goal of the next several sections is to establish that if a sample of a smooth surface Σ has the right properties, its restricted Delaunay triangulation is a triangulation of the surface. Our key tool for achieving this is an important result from computational topology called the Topological Ball Theorem, which states that the underlying space of $\operatorname{Del}|_\Sigma S$ is homeomorphic to Σ if every Voronoi k-face that intersects Σ does so transversally in a topological ball of dimension $k-1$. To state the theorem formally, we introduce the topological ball property.

Definition 13.3 (topological ball property)**.** Let $g \in \operatorname{Vor} S$ be a Voronoi face of dimension k, $0 \le k \le 3$. Let $g|_\Sigma$ be the restricted Voronoi face $g \cap \Sigma$. The face g satisfies the *topological ball property* (TBP) if $g|_\Sigma$ is empty or

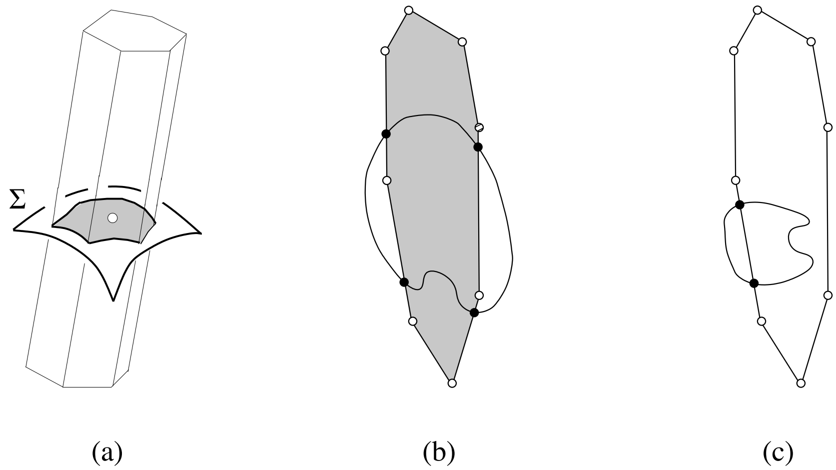


Figure 13.2: (a) Good: the intersection between Σ and each Voronoi cell or face is a topological ball of appropriate dimension. (b) Bad: the intersection between Σ and a Voronoi facet is not a single topological 1-ball. (c) Bad: the intersection between Σ and a Voronoi edge is not a single point.

(i) $g|_\Sigma$ is a topological $(k-1)$-ball (homeomorphic to $\mathbb{B}^{k-1}$), and

(ii) $(\mathrm{Int}\, g) \cap \Sigma = \mathrm{Int}\,(g|_\Sigma)$. Here we use the definition of "interior" for manifolds, Definition 12.12, not Definition 1.13.

The pair (S, Σ) satisfies the *TBP* if every Voronoi face $g \in \mathrm{Vor}\, S$ satisfies the *TBP*.

Figure 13.2 illustrates condition (i), which means that Σ intersects a Voronoi cell in a single topological disk, a Voronoi facet in a single topological 1-ball (a curve with two endpoints), a Voronoi edge in a single point, and a Voronoi vertex not at all. Condition (ii) rules out Σ intersecting the boundary of a Voronoi face without intersecting its interior infinitesimally close by. If all the intersections between Σ and Voronoi faces are transverse, condition (ii) is automatically satisfied. To prove that the TBP holds when S is sufficiently dense, we will explicitly prove that the Voronoi edges and Voronoi facets intersect Σ transversally. The fact that the Voronoi vertices do (i.e. no Voronoi vertex lies on Σ) follows because if a Voronoi vertex lay on Σ, then not all the Voronoi facets adjoining it could intersect Σ transversally in an interval. Trivially, all Voronoi 3-cells intersect Σ transversally.

Theorem 13.1 (Topological Ball Theorem)**.** *If the pair* (S, Σ) *satisfies the topological ball property, the underlying space of* $\mathrm{Del}|_\Sigma\, S$ *is homeomorphic to* Σ.

At first, this guarantee might not seem very impressive; after all, an elephant-shaped mesh of an airplane impresses nobody, even if the boundaries of both are topological spheres. The point, however, is that a subcomplex of $\mathrm{Del}\, S$ need not be a manifold without boundary, or even a manifold at all. $\mathrm{Del}|_\Sigma\, S$ can be a jumble of simplices. The Topological Ball Theorem will help us to guarantee that for a dense enough sample, $\mathrm{Del}|_\Sigma\, S$ is surprisingly well behaved. Sections 13.3–13.5 establish sampling conditions under which the TBP holds.

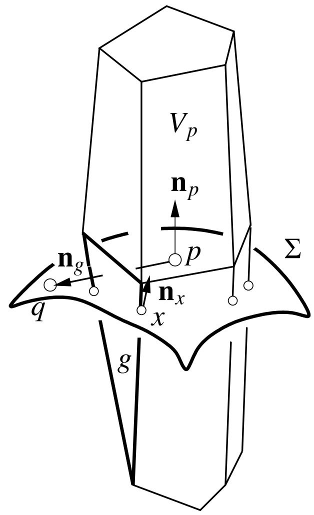


Figure 13.3: The normals $\mathbf{n}_p$ and $\mathbf{n}_x$ are almost orthogonal to $\mathbf{n}_g$.

13.3 Distances and angles in ε-samples

We begin by bounding the sizes of the restricted Delaunay edges and triangles of an ε-sample. If pq is a restricted Delaunay edge, it is dual to a Voronoi facet g that intersects Σ and lies on the bisector of pq. For any intersection point $x \in g \cap \Sigma$, the length of pq cannot be more than twice $d(p, x)$. Thus, if $d(p, x) \leq \varepsilon f(p)$, then $d(p, q) \leq 2\varepsilon f(p)$.

We can extend this argument to restricted Delaunay triangles too. A restricted Delaunay triangle τ is dual to a Voronoi edge τ^* that intersects Σ. An intersection point $x \in \tau^* \cap \Sigma$ is in every Voronoi cell having edge τ^*. If p is a vertex of τ, V_p is such a cell. The point x is the center of a circumball of the triangle τ. The circumradius of τ, being the radius of its smallest circumball, is at most $d(p, x)$. The following proposition is thus immediate.

Proposition 13.2. *For any $\varepsilon < 1$, the following properties hold.*

(i) *Let e be a restricted Delaunay edge with a vertex p. If the dual Voronoi facet of e intersects Σ in a point x such that $d(p, x) \leq \varepsilon f(p)$, the length of e is at most $2\varepsilon f(p)$.*

(ii) *Let τ be a restricted Delaunay triangle with a vertex p. If the dual Voronoi edge of τ intersects Σ in a point x such that $d(p, x) \leq \varepsilon f(p)$, the circumradius of τ is at most $\varepsilon f(p)$.*

In the previous chapter, we show that if edges and triangles connecting points on a smooth surface have small circumradii with respect to the local feature size, they lie nearly parallel to the surface. Hence, their dual Voronoi facets and edges intersect Σ almost orthogonally. The circumradius of a restricted Delaunay simplex is the distance from its vertices to the affine hull of its dual Voronoi face, so if the circumradius is small, the affine hull of the dual face intersects the surface near a sample point. The next two propositions quantify these statements.

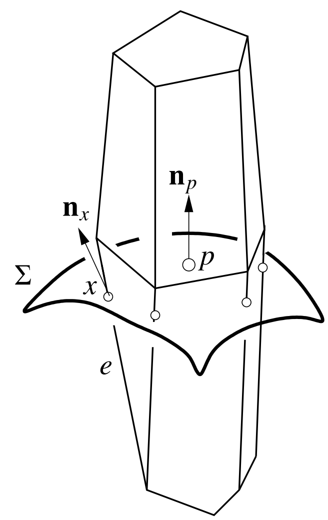


Figure 13.4: By Proposition 13.4, $\mathbf{n}_x$, e, and $\mathbf{n}_p$ are almost parallel.

Proposition 13.3. *Let g be a facet of a Voronoi cell V_p. Let $\Pi = \text{aff}\ g$. Let $\mathbf{n}_g$ be the normal to g. If there is an $\varepsilon < 1$ and a point $x \in \Pi \cap \Sigma$ such that $d(p, x) \leq \varepsilon f(p)$, then*

(i) $\angle_a(\mathbf{n}_g, \mathbf{n}_p) \geq \pi/2 - \arcsin \varepsilon$ *and*

(ii) $\angle_a(\mathbf{n}_g, \mathbf{n}_x) \geq \pi/2 - \arcsin \varepsilon - \alpha(\varepsilon)$*, which is greater than $\pi/6$ for $\varepsilon < 1/3$.*

Proof. Refer to Figure 13.3. Let pq be the Delaunay edge dual to g. By Proposition 13.2(i), $d(p, q) \leq 2\varepsilon f(p)$. Because pq is orthogonal to Π, $\angle_a(\mathbf{n}_g, \mathbf{n}_p) = \angle_a(pq, \mathbf{n}_p)$, and by the Edge Normal Lemma (Lemma 12.12), $\angle_a(pq, \mathbf{n}_p) \geq \pi/2 - \arcsin \varepsilon$, yielding (i).

By the Normal Variation Theorem (Theorem 12.8), $\angle(\mathbf{n}_p, \mathbf{n}_x) \leq \alpha(\varepsilon)$. By the triangle inequality,

$$\begin{aligned} \angle_a(\mathbf{n}_g, \mathbf{n}_x) &\geq \angle_a(\mathbf{n}_g, \mathbf{n}_p) - \angle_a(\mathbf{n}_p, \mathbf{n}_x) \\ &\geq \frac{\pi}{2} - \arcsin \varepsilon - \alpha(\varepsilon). \end{aligned}$$

□

The next proposition shows that if a Voronoi edge e intersects the surface Σ at a point close to the vertices of e's Delaunay dual triangle, then e is nearly orthogonal to Σ at that point, as illustrated in Figure 13.4. It is also nearly parallel to the normals at the dual triangle's vertices.

Proposition 13.4. *Let e be an edge of a Voronoi cell V_p. If there is a point $x \in e \cap \Sigma$ such that $d(p, x) \leq \varepsilon f(p)$ for some $\varepsilon < 1/2$, then*

(i) $\angle_a(e, \mathbf{n}_p) \leq \beta(\varepsilon)$ *and*

(ii) $\angle_a(e, \mathbf{n}_x) \leq \alpha(\varepsilon) + \beta(\varepsilon)$.

Proof. Let τ be the Delaunay triangle dual to e, and observe that e is parallel to the normal $\mathbf{n}_\tau$ to τ. Let q be the vertex of τ where the angle is largest. The circumradius of τ cannot exceed $d(p,x) \le \varepsilon f(p)$. Hence $d(p,q) \le 2\varepsilon f(p)$. By the Feature Translation Lemma (Lemma 12.2), $f(p) \le f(q)/(1-2\varepsilon)$. Therefore, the circumradius of τ is at most $\frac{\varepsilon}{1-2\varepsilon} f(q)$. We apply the Triangle Normal Lemma (Lemma 12.14) to τ to obtain

$$\angle_a(\mathbf{n}_\tau, \mathbf{n}_q) \le \arcsin \frac{\varepsilon}{1-2\varepsilon} + \arcsin\left(\frac{2}{\sqrt{3}} \sin\left(2 \arcsin \frac{\varepsilon}{1-2\varepsilon}\right)\right).$$

Because $2\varepsilon < 1$ by assumption, the Normal Variation Theorem (Theorem 12.8) applies, giving

$$\angle(\mathbf{n}_p, \mathbf{n}_q) \le \alpha(2\varepsilon) \quad \text{and} \quad \angle(\mathbf{n}_p, \mathbf{n}_x) \le \alpha(\varepsilon).$$

Therefore,

$$\begin{aligned} \angle_a(e, \mathbf{n}_p) &\le \angle_a(\mathbf{n}_p, \mathbf{n}_q) + \angle_a(e, \mathbf{n}_q) \\ &\le \alpha(2\varepsilon) + \angle_a(\mathbf{n}_\tau, \mathbf{n}_q) \\ &= \beta(\varepsilon), \end{aligned}$$

proving (i). The correctness of (ii) follows by the triangle inequality: $\angle_a(e, \mathbf{n}_x) \le \angle_a(\mathbf{n}_p, \mathbf{n}_x) + \angle_a(e, \mathbf{n}_p) \le \alpha(\varepsilon) + \beta(\varepsilon)$. □

The next proposition proves a fact about distances given an angle constraint. It is similar to the Long Distance Lemma (Lemma 12.13), which is used in the proof.

Proposition 13.5. *Let p, x, and y be three points on Σ. If there is an $\varepsilon \le 0.15$ such that $\angle_a(xy, \mathbf{n}_x) \le \alpha(\varepsilon) + \beta(\varepsilon)$, then at least one of $d(p,x)$ or $d(p,y)$ is larger than $\varepsilon f(p)$.*

Proof. Assume to the contrary that both $d(p,x)$ and $d(p,y)$ are at most $\varepsilon f(p)$. By the Feature Translation Lemma (Lemma 12.2), $f(x) \le f(p)/(1-\varepsilon)$. By the Three Points Lemma (Lemma 12.3),

$$d(x,y) \le 2\varepsilon f(p) \le \frac{2\varepsilon}{1-\varepsilon} f(x) = 2\alpha(\varepsilon) f(x).$$

On the other hand, by the Long Distance Lemma (Lemma 12.13),

$$d(x,y) \ge 2f(x) \cos \angle(xy, \mathbf{n}_x) \ge 2f(x) \cos(\alpha(\varepsilon) + \beta(\varepsilon)).$$

The lower and upper bounds on $d(x,y)$ together yield

$$\alpha(\varepsilon) \ge \cos(\alpha(\varepsilon) + \beta(\varepsilon)).$$

This contradicts condition A.1, which is satisfied for $\varepsilon \le 0.15$. □

13.4 Local properties of restricted Voronoi faces

The goal of this section is to prove that a restricted Voronoi face has a simple topology if it has a vertex (i.e. a restricted Voronoi vertex) and all its vertices are close to a sample point.

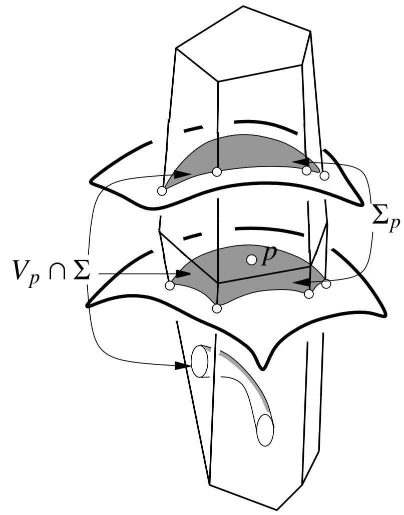


Figure 13.5: The large polyhedron is V_p, and the three curved components inside it are $V_p \cap \Sigma$. Σ_p consists of the two shaded components, which intersect Voronoi edges. The cylindrical component at the bottom is not part of Σ_p, because it does not intersect any Voronoi edge.

By "simple topology," we mean that Σ intersects a Voronoi edge in a single point, a Voronoi facet in a single curve that is not a loop, and a Voronoi cell in a single topological disk. We prove these three cases separately in results called the Voronoi Edge Lemma (Lemma 13.7), the Voronoi Facet Lemma (Lemma 13.9), and the Voronoi Cell Lemma (Lemma 13.12). We collect these results together as the Small Intersection Theorem (Theorem 13.6).

The main difficulty to overcome is that a Voronoi cell V_p can intersect the surface Σ in multiple connected components. The characterization of the results above is not quite correct, because they apply only to the connected components that adjoin a restricted Voronoi vertex. We introduce a notation for these components because they play an important role in the forthcoming analysis.

Definition 13.4 (ε-small)**.** Let Σ_p be the union of the connected components of $V_p \cap \Sigma$ that adjoin at least one restricted Voronoi vertex. We say that Σ_p is *ε-small* if Σ_p is empty or the distances from p to every restricted Voronoi vertex in Σ_p are less than $\varepsilon f(p)$.

Figure 13.5 shows Σ_p for a sample point p. Observe that Σ_p is a subset of p's restricted Voronoi cell $V_p|_\Sigma$. The Small Intersection Theorem formalizes the properties of the intersections between Σ_p and the faces of V_p when Σ_p is 0.09-small. The connected components of a 1-manifold are either *loops*, that is, connected curves with no boundary, or *topological intervals*, also known as 1-balls: curves having two boundary points each.

Theorem 13.6 (Small Intersection Theorem)**.** *If there is an $\varepsilon \leq 0.09$ such that Σ_p is nonempty and ε-small, then the following properties hold.*

(i) *Σ_p is a topological disk and every point in Σ_p is a distance less than $\varepsilon f(p)$ from p.*

(ii) *Every edge of V_p that intersects Σ does so transversally in a single point.*

(iii) *Every facet g of V_p that intersects Σ_p does so in a topological interval, and the intersection is transverse at every point of $g \cap \Sigma_p$.*

Proof. (i) Follows directly from the Voronoi Cell Lemma (Lemma 13.12). (ii) Follows directly from the Voronoi Edge Lemma (Lemma 13.7). (iii) Because of (i), Σ_p has a single boundary and it intersects a Voronoi edge. Thus, the boundary of Σ_p cannot lie completely in the interior of any facet of V_p without intersecting one of its edges. Then, the Voronoi Facet Lemma (Lemma 13.9) applies. □

The three lemmas mentioned above occupy the remainder of this section. The first of them is the easiest to prove.

Lemma 13.7 (Voronoi Edge Lemma)**.** *Suppose that Σ intersects an edge e of a Voronoi cell V_p. If there is an $\varepsilon \leq 0.15$ such that $d(p, x) \leq \varepsilon f(p)$ for every point $x \in e \cap \Sigma$, then e intersects Σ transversally in a single point.*

Proof. Let x be a point in $e \cap \Sigma$. By Proposition 13.4(ii), we have

$$\angle_a(e, \mathbf{n}_x) \leq \alpha(\varepsilon) + \beta(\varepsilon).$$

Assume for the sake of contradiction that e does not intersect Σ transversally in a single point. Either e intersects Σ tangentially at x or there is a point $y \in e \cap \Sigma$ other than x; Figure 13.6 shows both cases. In the first case, $\angle_a(e, \mathbf{n}_x) = \pi/2$, which implies that $\alpha(\varepsilon) + \beta(\varepsilon) \geq \pi/2$, which is impossible for any $\varepsilon \leq 0.15$. In the second case, $\angle_a(xy, \mathbf{n}_x) = \angle_a(e, \mathbf{n}_x) \leq \alpha(\varepsilon) + \beta(\varepsilon)$. But then Proposition 13.5 states that $d(p, x)$ or $d(p, y)$ is larger than $\varepsilon f(p)$, contradicting the assumption. □

We use the next proposition to prove the Voronoi Facet Lemma (Lemma 13.9) and the Voronoi Cell Lemma (Lemma 13.12). It says that if the intersection of a Voronoi facet with Σ includes a curve with nonempty boundary whose boundary points are close to a sample point, then the entire curve is close to the sample point.

Proposition 13.8. *Let g be a facet of a Voronoi cell V_p. Suppose that $g \cap \Sigma$ contains a topological interval I. If there is an $\varepsilon \leq 0.15$ such that the distances from p to the endpoints of I are less than $\varepsilon f(p)$, then the distance from p to any point in I is less than $\varepsilon f(p)$.*

Proof. Let B be the Euclidean 3-ball centered at p with radius $\varepsilon f(p)$. Let $\Pi = \operatorname{aff} g$. As the distances from p to the endpoints of I are less than $\varepsilon f(p)$, the endpoints of I lie in $\operatorname{Int} D$, where D is the Euclidean disk $B \cap \Pi$. To prove the proposition, it suffices to show that $I \subseteq \operatorname{Int} D$. Assume to the contrary that $I \not\subseteq \operatorname{Int} D$. Refer to Figure 13.7.

Let C be the connected component of $\Pi \cap \Sigma$ containing I. Observe that $C \not\subseteq \operatorname{Int} D$ because $I \subset C$ and $I \not\subseteq \operatorname{Int} D$ by assumption. For any point $x \in \Pi \cap \Sigma \cap D$, because $d(p, x) \leq \varepsilon f(p)$, the point x cannot be a tangential contact point between Π and Σ as that would contradict Proposition 13.3(ii). Thus, $C \cap \operatorname{Int} D$ is a collection of disjoint topological intervals.

We claim that $C \cap \operatorname{Int} D$ consists of at least two topological intervals. Assume to the contrary that $C \cap \operatorname{Int} D$ is one topological interval. Recall that $I \subset C$, $I \not\subseteq \operatorname{Int} D$, and the

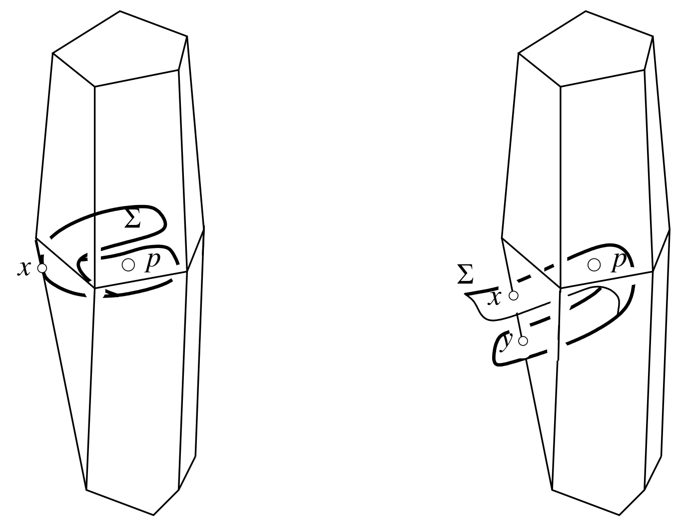


Figure 13.6: (a) A Voronoi edge intersects Σ tangentially at a single point. (b) A Voronoi edge intersects Σ in two points.

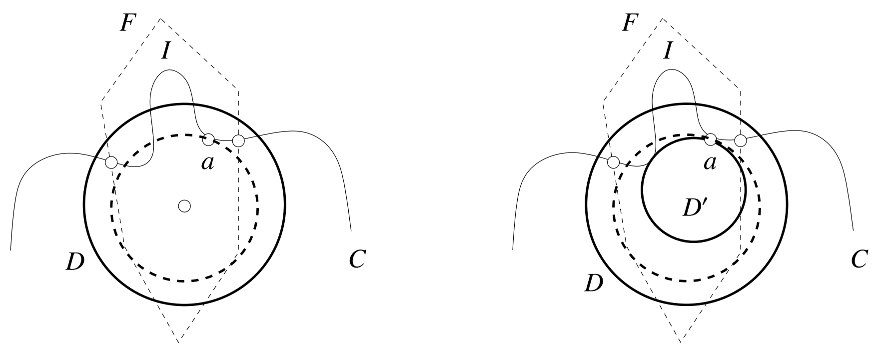


Figure 13.7: (a) D is shrunk radially until it is tangent to C at some point a. (b) The shrinking of D is continued by moving the center toward a.

endpoints of I lie in $C \cap \operatorname{Int} D$. It follows that $(C \cap \operatorname{Int} D) \cup I$ is a loop. Take an edge e of g that contains an endpoint x of I. Since Σ meets e transversally, the affine hull ℓ of e crosses $C \cap \operatorname{Int} D$ at x. Since g is convex, it lies on one side of ℓ. After $C \cap \operatorname{Int} D$ leaves the side of ℓ containing g, it must return to g in order to form a loop with I, which lies in g. It means that $C \cap \operatorname{Int} D$ has to cross ℓ at least twice. Hence, ℓ intersects $C \cap \operatorname{Int} D$ at x and another point y. Since $x, y \in D$, both $d(p, x)$ and $d(p, y)$ are at most $\varepsilon f(p)$. By Proposition 13.4, $\angle_a(xy, \mathbf{n}_x) = \angle_a(e, \mathbf{n}_x) \leq \alpha(\varepsilon) + \beta(\varepsilon)$. But then Proposition 13.5 implies that $d(p, x)$ or $d(p, y)$ is larger than $\varepsilon f(p)$, a contradiction.

So we can assume that $C \cap \operatorname{Int} D$ consists of at least two disjoint topological intervals.

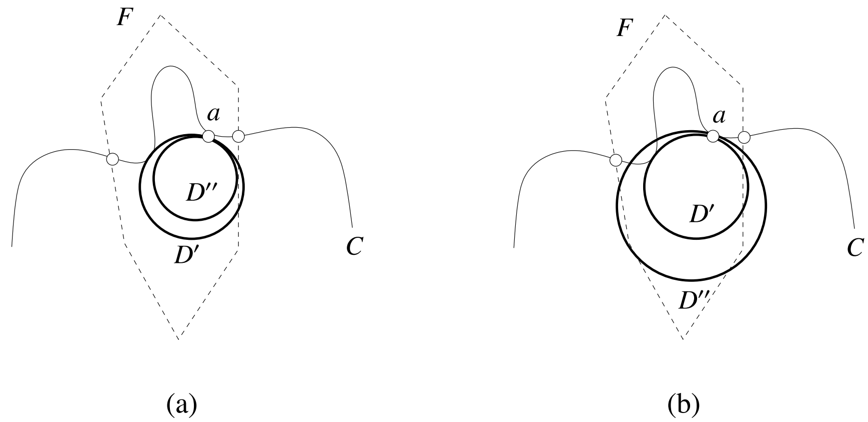


Figure 13.8: D' and D''.

Then, D can be shrunk to a smaller disk D' as follows so that D' meets C tangentially at two points and $C \cap \operatorname{Int} D' = \emptyset$. First, shrink D radially until it meets C tangentially at some point a. Refer to Figure 13.7(a). It follows that $d(p, a) < \varepsilon f(p)$. If this shrunk D does not meet the requirement of D' yet, shrink it further by moving its center toward a until a disk D' is obtained as required. Refer to Figure 13.7(b). Observe that a is one of the contact points between D' and C.

The affine hull Π of g intersects the two medial balls of Σ at a in two disks. Among these two disks, let D'' be the one that intersects D'. Let B'' be the medial ball such that $D'' = B'' \cap \Pi$. The boundaries of D' and D'' meet tangentially at a. So either $D'' \subseteq D'$ (Figure 13.8(a)) or $D' \subset D''$ (Figure 13.8(b)).

We claim that $D'' \subseteq D'$ and the radius of D'' is greater than $\varepsilon f(p)$. Suppose that $D' \subset D''$. By construction, D' meets Σ tangentially at two points. So one of these contact points must lie in $\operatorname{Int} D''$. This is a contradiction because $D'' = B'' \cap \Pi$ and $\operatorname{Int} B'' \cap \Sigma = \emptyset$ as B'' is a medial ball. This shows that $D'' \subseteq D'$. By Proposition 13.3(ii), the acute angle between Π and $\mathbf{n}_a$ is at most $\alpha(\varepsilon) + \arcsin \varepsilon$. The angle between the diametric segments of B'' and D'' adjoining a is equal to the angle between $\mathbf{n}_a$ and Π. Therefore,

$$\begin{aligned} \operatorname{radius}(D'') &\geq \operatorname{radius}(B'') \cdot \cos(\alpha(\varepsilon) + \arcsin \varepsilon) \\ &\geq f(a) \cdot \cos(\alpha(\varepsilon) + \arcsin \varepsilon). \end{aligned}$$

Observe that $\beta(\varepsilon) > \arcsin \varepsilon$. Also, $\cos(\alpha(\varepsilon) + \beta(\varepsilon)) > \alpha(\varepsilon)$ by condition A.1 as $\varepsilon \leq 0.15$. Thus,

$$\begin{aligned} \operatorname{radius}(D'') &\geq f(a) \cdot \cos(\alpha(\varepsilon) + \arcsin \varepsilon) \\ &> f(a) \cdot \cos(\alpha(\varepsilon) + \beta(\varepsilon)) \\ &> \alpha(\varepsilon) \cdot f(a) \\ &= \frac{\varepsilon}{1 - \varepsilon} f(a). \end{aligned}$$

It follows from the Feature Translation Lemma (Lemma 12.2) that $\operatorname{radius}(D'') > \varepsilon f(p)$. This completes the proof of our claim that $D'' \subseteq D'$ and $\operatorname{radius}(D'') > \varepsilon f(p)$; therefore, radius($D'$) $> \varepsilon f(p)$. But D' is obtained by shrinking $D = B \cap \Pi$ where radius(B) $= \varepsilon f(p)$. We reach a contradiction. In all, the contrapositive assumption that $I \not\subseteq \operatorname{Int} D$ cannot hold. It follows that the distance from p to any point in I is less than $\varepsilon f(p)$. $\square$

With this preparation, we prove the Voronoi Facet Lemma: the intersection between a Voronoi facet and Σ is a single topological interval if each component of the intersection contains at least one restricted Voronoi vertex and every such restricted Voronoi vertex is close to a sample point.

Lemma 13.9 (Voronoi Facet Lemma)**.** *Let g be a facet of a Voronoi cell V_p. Let C_g be the union of the curves in $g \cap \Sigma$ that each contain a restricted Voronoi vertex. If there is an $\varepsilon \leq 0.09$ such that every restricted Voronoi vertex in* $\operatorname{Bd} g$ *is at a distance less than $\varepsilon f(p)$ from p, then C_g consists of exactly one topological interval at which g intersects Σ transversally.*

Proof. First we claim that C_g includes no loop. If there is one, it must intersect a Voronoi edge by the assumption that it contains a restricted Voronoi vertex. Since the distance from p to every restricted Voronoi vertex in C_g is less than $\varepsilon f(p)$, the affine hull of g intersects Σ transversally at each such vertex by Proposition 13.3(ii). Thus, a loop in C_g can only meet a Voronoi edge tangentially. But this would mean that the edge meets Σ tangentially, which is forbidden by the Voronoi Edge Lemma (Lemma 13.7) for $\varepsilon \leq 0.09$. So C_g includes only topological intervals.

Assume for the sake of contradiction that there are two or more intervals and let I and I' be any two of them. Let u and v be the endpoints of I. Let x and y be the endpoints of I'. By the Voronoi Edge Lemma, no edge of g intersects Σ in two or more points, so the four edges of g containing u, v, x, and y are distinct. Let Q be the convex quadrilateral on aff g bounded by the affine hulls of these four edges. We call Q's edges e_u, e_v, e_x, and e_y according to the interval endpoints that they contain. Refer to Figure 13.9(a).

The distances $d(p,u)$, $d(p,v)$, $d(p,x)$, and $d(p,y)$ are less than $\varepsilon f(p)$ by assumption. Consider the Delaunay triangles dual to the edges of g containing u, v, x, and y. Their circumradii are at most $\varepsilon f(p)$ by Proposition 13.2(ii). By Proposition 13.4(i), the angles $\angle_a(e_u, \mathbf{n}_p)$, $\angle_a(e_v, \mathbf{n}_p)$, $\angle_a(e_x, \mathbf{n}_p)$, and $\angle_a(e_y, \mathbf{n}_p)$ are at most $\beta(\varepsilon)$. By Proposition 13.3(i), the acute angle between aff g and $\mathbf{n}_p$ is at most $\arcsin \varepsilon$. Let $\tilde{\mathbf{n}}_p$ be the projection of $\mathbf{n}_p$ onto aff g. So $\angle_a(\mathbf{n}_p, \tilde{\mathbf{n}}_p) \leq \arcsin \varepsilon$. It follows that

$$\angle_a(e_u, \tilde{\mathbf{n}}_p) \leq \angle_a(e_u, \mathbf{n}_p) + \angle_a(\mathbf{n}_p, \tilde{\mathbf{n}}_p) \leq \beta(\varepsilon) + \arcsin \varepsilon.$$

Similarly, the angles $\angle_a(e_v, \tilde{\mathbf{n}}_p)$, $\angle_a(e_x, \tilde{\mathbf{n}}_p)$, and $\angle_a(e_y, \tilde{\mathbf{n}}_p)$ are at most $\beta(\varepsilon) + \arcsin \varepsilon$. The convexity of Q implies that one of its interior angles must be at least $\pi - 2(\beta(\varepsilon) + \arcsin \varepsilon)$, say, the interior angle between e_v and e_x. In this case, a line parallel to $\tilde{\mathbf{n}}_p$ may either cut through or be tangent to the corner of Q between e_v and e_x. Figures 13.9(b) and 13.9(c) illustrate these two possibilities. In both configurations, $\angle_a(vx, e_v) \leq \beta(\varepsilon) + \arcsin \varepsilon$ or $\angle_a(vx, e_x) \leq \beta(\varepsilon) + \arcsin \varepsilon$. Without loss of generality, assume that $\angle_a(vx, e_x) \leq \beta(\varepsilon) + \arcsin \varepsilon$.

By Proposition 13.4(ii), $\angle_a(e_x, \mathbf{n}_x) \leq \alpha(\varepsilon) + \beta(\varepsilon)$. Then

$$\begin{aligned}\angle_a(vx, \mathbf{n}_x) &\leq \angle_a(vx, e_x) + \angle_a(e_x, \mathbf{n}_x) \\ &\leq \alpha(\varepsilon) + \arcsin \varepsilon + 2\beta(\varepsilon).\end{aligned}$$

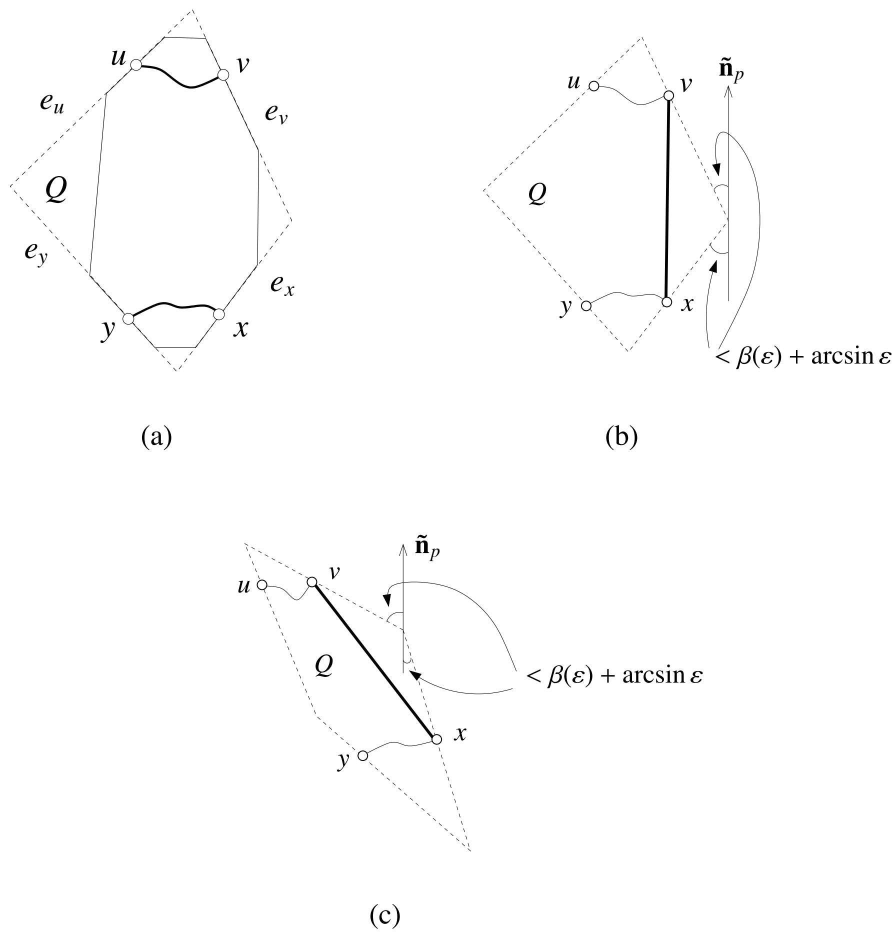


Figure 13.9: A Voronoi facet (bounded by solid line segments) intersects Σ in two topological intervals (shown as curves). The convex quadrilateral Q is bounded by dashed line segments.

On the other hand, $d(v, x) \leq 2\varepsilon f(x)/(1 - \varepsilon)$ by the Three Points Lemma (Lemma 12.3). Then, the Edge Normal Lemma (Lemma 12.12) implies that

$$\begin{aligned} \angle_a(vx, \mathbf{n}_x) &\geq \frac{\pi}{2} - \arcsin \frac{\varepsilon}{1-\varepsilon} \\ &= \frac{\pi}{2} - \arcsin \alpha(\varepsilon) \\ &= \arccos \alpha(\varepsilon). \end{aligned}$$

The upper and lower bounds on $\angle_a(vx, \mathbf{n}_x)$ together give

$$\begin{aligned} & \arccos \alpha(\varepsilon) \leq \alpha(\varepsilon) + \arcsin \varepsilon + 2\beta(\varepsilon) \\ \Rightarrow \quad & \alpha(\varepsilon) \geq \cos(\alpha(\varepsilon) + \arcsin \varepsilon + 2\beta(\varepsilon)). \end{aligned}$$

This contradicts condition A.2, which is satisfied for $\varepsilon \leq 0.09$. Therefore, it contradicts the assumption that C_g includes more than one topological interval.

It remains to prove that g intersects Σ transversally at C_g. Suppose to the contrary Σ intersects g tangentially at some point s in C_g. It follows from Proposition 13.8 that $d(p,s) < \varepsilon f(p)$. Therefore, $\angle_a(\mathbf{n}_p, \mathbf{n}_s) \le \alpha(\varepsilon)$ by the Normal Variation Theorem (Theorem 12.8). The triangle inequality and Proposition 13.3(i) give

$$\begin{aligned} \angle_a(\mathbf{n}_g, \mathbf{n}_s) &\ge \angle_a(\mathbf{n}_g, \mathbf{n}_p) - \angle_a(\mathbf{n}_p, \mathbf{n}_s) \\ &\ge \frac{\pi}{2} - \arcsin\varepsilon - \alpha(\varepsilon). \end{aligned}$$

The quantity $\pi/2 - \arcsin\varepsilon - \alpha(\varepsilon)$ is positive for $\varepsilon \le 0.09$. We reach a contradiction because $\angle_a(\mathbf{n}_g, \mathbf{n}_s)$ should be zero as Σ is tangent to g at s. $\square$

Recall the surface Σ_p from Definition 13.4. If Σ_p is ε-small, the Voronoi Edge and Voronoi Facet Lemmas state that the edges and facets of V_p intersect Σ transversally, so Σ_p is a 2-manifold whose boundary is a 1-manifold without boundary, i.e. a collection of loops. Classify these boundary loops into two types.

- Type 1: A loop that intersects one or more Voronoi edges. The intersection points are restricted Voronoi vertices.
- Type 2: A loop that does not intersect any Voronoi edge.

Although every connected component of Σ_p adjoins a restricted Voronoi vertex by the definition of Σ_p, a connected component can have more than one boundary loop (e.g. if it is homeomorphic to an annulus), only one of which must be of Type 1.

The forthcoming Voronoi Cell Lemma says that for a sufficiently small ε, Σ_p is a topological disk. This requires us to argue that Σ_p has only one boundary loop.

We show some properties of Type 1 loops in Proposition 13.10 below. The result following it, Proposition 13.11, shows that Σ_p has exactly one Type 1 loop.

Proposition 13.10. *Let Σ_p be ε-small for some $\varepsilon \le 0.09$. Let C be a loop of Type 1 in* $\mathrm{Bd}\,\Sigma_p$. *Then, C bounds a topological disk $D \subset \Sigma$ such that $d(p,x) < \varepsilon f(p)$ for every point $x \in D$. Furthermore, if D does not include any other Type 1 loop, then*

(i) *D does not include any other loop of* $\mathrm{Bd}\,\Sigma_p$,

(ii) *D is included in V_p, and*

(iii) *D is a connected component of Σ_p.*

Proof. Proposition 13.8 and the definition of Type 1 loop imply that there is a Voronoi facet of V_p that contains a single topological interval of each Type 1 loop. Continuing the same argument for other facets on which the two end points of the topological interval lie, we conclude that each Type 1 loop consists of a set of topological intervals residing in Voronoi facets of V_p all of whose points are a distance less than $\varepsilon f(p)$ away from p. It follows that all Type 1 loops lie strictly inside a Euclidean 3-ball $B = B(p, \varepsilon f(p))$. The Voronoi Edge Lemma (Lemma 13.7) and the Voronoi Facet Lemma (Lemma 13.9) show that Σ intersects $\mathrm{Bd}\,V_p$ transversally at these loops, so the intersection is a 1-manifold. By the Small Ball Lemma (Lemma 12.7), $B \cap \Sigma$ is a topological disk. It follows that each Type 1

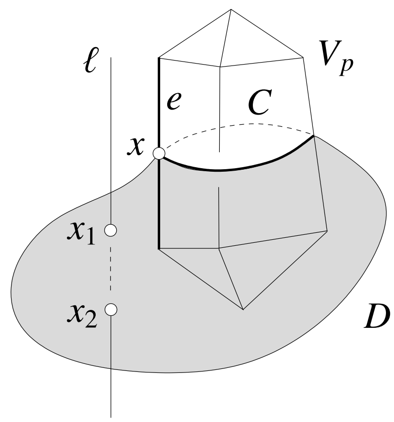


Figure 13.10: Proof of Proposition 13.10: disk D with the opening C is like a "sack" that encloses a part of V_p.

loop bounds a topological disk in $B \cap \Sigma$ that lies strictly inside B. This proves the first part of the proposition: C bounds a topological disk $D \subset \Sigma$ and $d(p, x) < \varepsilon f(p)$ for every point $x \in D$.

Consider (i). By assumption, the topological disk D does not include other loops of Type 1. Assume to the contrary that D includes a loop C' of Type 2. So C' is contained in a facet, say g, of V_p. Since D lies strictly inside B, so does C'. Take any point $x \in C'$. Consider the line ℓ_x through x parallel to the projection of $\mathbf{n}_x$ onto g. By Proposition 13.3(ii) $\angle_a(\ell_x, \mathbf{n}_x) \leq \alpha(\varepsilon) + \arcsin \varepsilon \leq \pi/3$. It means that ℓ_x intersects C' transversally and thus intersects it at another point x'. Both $d(p, x)$ and $d(p, x')$ are less than $\varepsilon f(p)$. Also, $\angle_a(\ell_x, \mathbf{n}_x) \leq \alpha(\varepsilon) + \arcsin \varepsilon \leq \alpha(\varepsilon) + \beta(\varepsilon)$. Therefore, by Proposition 13.5, $d(p, x)$ or $d(p, x')$ is greater than $\varepsilon f(p)$, a contradiction.

Consider (ii). Because V_p is a closed set, it is sufficient to show that $\operatorname{Int} D$ is included in V_p. Suppose not. Then, $\operatorname{Int} D$ must lie completely outside V_p; otherwise, $\operatorname{Int} D$ would include a boundary loop of Σ_p, which is prohibited by (i). Refer to Figure 13.10. Let e be an edge of V_p that intersects C. Let x be an intersection point of e and C. By Proposition 13.4(ii), $\angle_a(e, \mathbf{n}_x) \leq \alpha(\varepsilon) + \beta(\varepsilon)$, which is less than $\pi/3$ for $\varepsilon \leq 0.09$. Thus, $\operatorname{aff} e$ intersects Σ transversally at x. Let ℓ be a line outside V_p that is parallel to and arbitrarily close to $\operatorname{aff} e$. Then ℓ must intersect $\operatorname{Int} D$ transversally at a point x_1 arbitrarily close to x.

As $\operatorname{Bd} V_p$ is a topological sphere, C cuts it into two topological disks. Let T be one of them. The union $T \cup D$ is a topological sphere. Because ℓ intersects $\operatorname{Int} D$ at x_1, ℓ must intersect $T \cup D$ at another point $x_2 \neq x_1$.

The point x_2 must lie in D because $T \subseteq \operatorname{Bd} V_p$ and ℓ lies outside V_p. By the Long Distance Lemma (Lemma 12.13), $d(x_1, x_2) \geq 2f(x_1)\cos(\angle_a(\ell, \mathbf{n}_{x_1}))$. Observe that x_1 is arbitrarily close to x and $\angle_a(\ell, \mathbf{n}_x) = \angle_a(e, \mathbf{n}_x) < \pi/3$. Thus, $\angle_a(\ell, \mathbf{n}_{x_1}) < \pi/3$ and so $d(x_1, x_2) > 2f(x_1)\cos \pi/3 = f(x_1)$. As x_1 is arbitrarily close to x and f is continuous, $d(x_1, x_2) \geq f(x)$; thus, $d(x_1, x_2) \geq (1 - \varepsilon) f(p)$ by the Feature Translation Lemma (Lemma 12.2). But this is a contradiction because D lies inside B and B has diameter $2\varepsilon f(p) < (1 - \varepsilon) f(p)$ for $\varepsilon \leq 1/3$.

The correctness of (iii) follows immediately from (i) and (ii). □

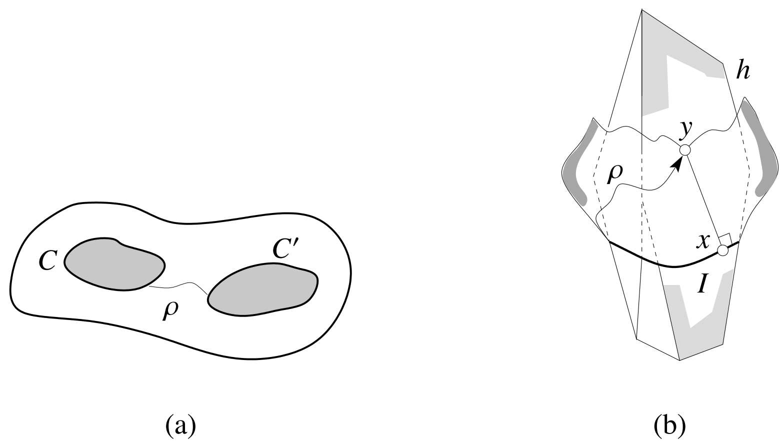


Figure 13.11: (a) Two loops C and C' drawn schematically on the patch $B \cap \Sigma$. The path ρ starting from C goes outside V_p and then has to reach V_p again to reach C'. (b) A different view with the polytope P. The lower bold curve denotes C, whose intersection with the shaded facet h is a topological interval I. The curved patch shown is part of $(B \cap \Sigma) \setminus \mathrm{Int}\, V_p$. The curved path on it is ρ.

We use Proposition 13.10 to show that Σ_p has only one Type 1 boundary loop if Σ_p is ε-small for $\varepsilon \leq 0.09$, which will prepare us to prove the Voronoi Cell Lemma.

Proposition 13.11. *If Σ_p is ε-small for some $\varepsilon \leq 0.09$, then $\mathrm{Bd}\, \Sigma_p$ has exactly one Type 1 loop.*

Proof. The definition of Σ_p implies that its boundary has at least one Type 1 loop. Each Type 1 loop bounds a topological disk in Σ by Proposition 13.10. Because the loops are disjoint, the topological disks bounded by them are either disjoint or nested. So there is a loop C of Type 1 bounding a topological disk D in Σ such that D contains no Type 1 loop. By Proposition 13.10(iii), D is a connected component of Σ_p.

If $\mathrm{Bd}\, \Sigma_p$ does not have any Type 1 loop other than C, there is nothing to prove, so assume to the contrary that there is another loop C' of Type 1 in $\mathrm{Bd}\, \Sigma_p$. Consider the set of facets of V_p that intersect C. Each facet in this set bounds a halfspace containing p. The intersection of these halfspaces is a convex polytope P that includes V_p. As D lies in V_p, it lies in P too.

Let $B = B(p, \varepsilon f(p))$. By Proposition 13.8, both loops C and C' lie inside $B \cap \Sigma$. Let ρ be a curve in $(B \cap \Sigma) \setminus \mathrm{Int}\, V_p$ that connects C to C'. Since $D \subset P$ and the contact between D and $\mathrm{Bd}\, P$ is not tangential, ρ leaves P where ρ leaves D. Since $C' \subset V_p \subseteq P$, the path ρ must return to some facet of P to meet C'. Let h be a facet of P that ρ intersects after leaving D. Let y be a point in $\rho \cap h$. Let g be the facet of V_p included in h. By the definition of P, C must intersect g.

The Voronoi Facet Lemma (Lemma 13.9) implies that $C \cap g$ is one topological interval. Call this topological interval I. Refer to Figure 13.11.

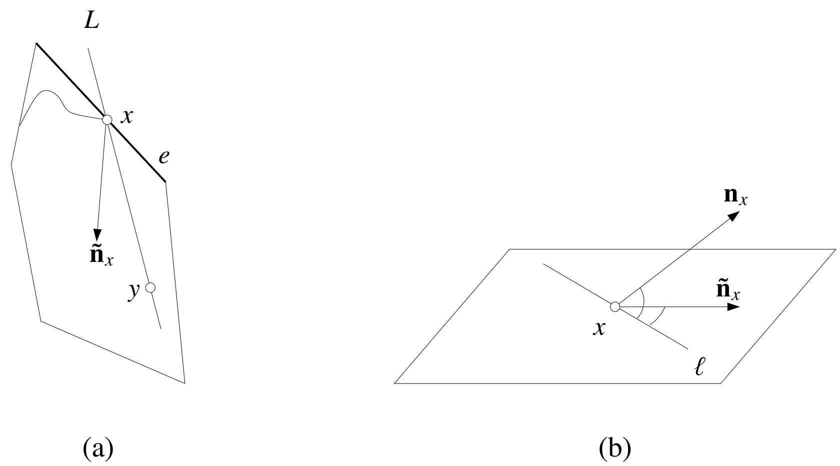


Figure 13.12: (a) The curve is $C \cap g$ and $\angle_a(L, \tilde{\mathbf{n}}_x) \leq \angle_a(e, \tilde{\mathbf{n}}_x)$. (b) The angle $\angle_a(\ell, \mathbf{n}_x)$ is an increasing function of $\angle_a(\ell, \tilde{\mathbf{n}}_x)$.

Claim 1. *Every edge of g that contains an endpoint of I is included in some edge of h.*

Proof. Consider an endpoint z of I. The point z lies on the boundary of g, which means that the other facet(s) of V_p that share z with g are intersected by C. So the affine hulls of these facets bound P. It follows that the edges of g containing z are included in some edges of h. □

By Claim 1, the endpoints of I lie on the boundary of h, and hence, $C \cap h = C \cap g = I$. Let x be the point closest to y on I. Then $x \in I \subset C \subset B \cap \Sigma$ and $y \in \rho \subset B \cap \Sigma$. So the distances from p to x and y are at most $\text{radius}(B) = \varepsilon f(p)$. Let L be the line through x and y. There are two cases to consider.

- Case 1: x lies in the interior of h. Then L intersects I at x at a right angle. This means that L is the projection of $\mathbf{n}_x$ onto aff h. By Proposition 13.3(ii), $\angle_a(L, \mathbf{n}_x) \leq \alpha(\varepsilon) + \arcsin \varepsilon \leq \alpha(\varepsilon) + \beta(\varepsilon)$. But then $d(p, x)$ or $d(p, y)$ is greater than $\varepsilon f(p)$ by Proposition 13.5, a contradiction.

- Case 2: x lies on the boundary of h. Let e be an edge of h containing x. By Claim 1, e includes an edge of g that contains x. Let $\tilde{\mathbf{n}}_x$ be the projection of $\mathbf{n}_x$ onto aff h. Refer to Figure 13.12(a). Because x is the point closest to y on I, $\angle_a(L, \tilde{\mathbf{n}}_x) \leq \angle_a(e, \tilde{\mathbf{n}}_x)$. For any line ℓ in aff h through x, the angle $\angle_a(\ell, \mathbf{n}_x)$ increases as the angle $\angle(\ell, \tilde{\mathbf{n}}_x)$ increases. Refer to Figure 13.12(b). We conclude that $\angle_a(L, \mathbf{n}_x) \leq \angle(e, \mathbf{n}_x)$, which is at most $\alpha(\varepsilon) + \beta(\varepsilon)$ by Proposition 13.4(ii). But then $d(p, x)$ or $d(p, y)$ is greater than $\varepsilon f(p)$ by Proposition 13.5, a contradiction.

It follows from the contradiction that $\text{Bd}\, \Sigma_p$ includes only one Type 1 loop. □

We now have all the ingredients to prove the Voronoi Cell Lemma.

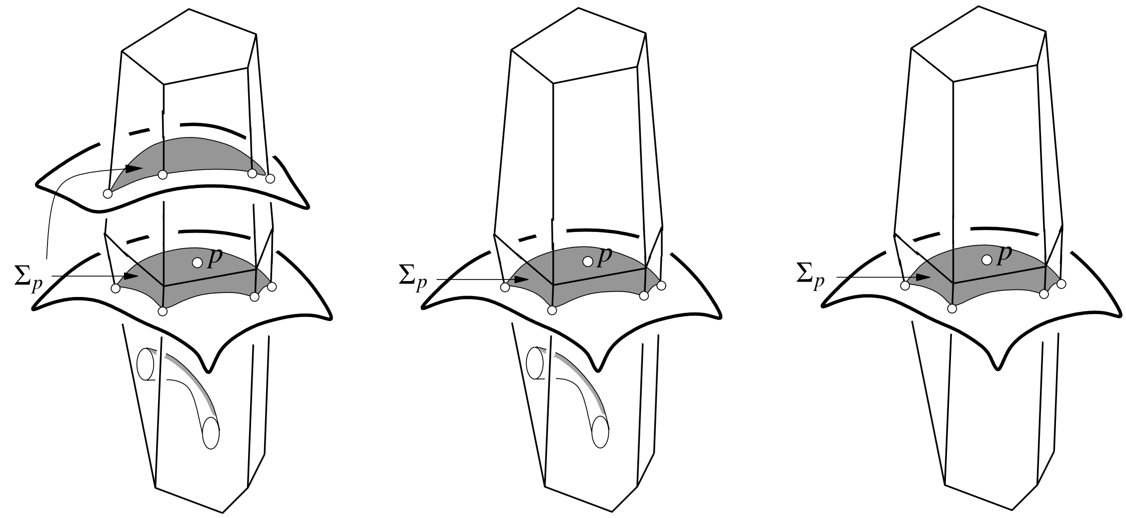


Figure 13.13: Let $\varepsilon \le 0.09$. If Σ_p is ε-small, the configuration at left is impossible by the Small Intersection Theorem, but the configuration at center is possible because the small cylindrical component is not part of Σ_p. If Σ_p is ε-small for *every* $p \in S$ and at least one Voronoi edge intersects Σ, the configuration at center is impossible by the Voronoi Intersection Theorem (Theorem 13.14), and only the configuration at right is possible.

Lemma 13.12 (Voronoi Cell Lemma)**.** *If Σ_p is nonempty and ε-small for some $\varepsilon \le 0.09$, then Σ_p is a topological disk and $d(p, x) < \varepsilon f(p)$ for every point $x \in \Sigma_p$.*

Proof. By Propositions 13.10 and 13.11, Σ_p has exactly one boundary loop C of Type 1 and C bounds a topological disk D, which is a connected component of Σ_p. There is no other connected component in Σ_p because, by the definition of Σ_p, such a component would have another Type 1 boundary loop, contradicting Proposition 13.11. Hence, $D = \Sigma_p$. By Proposition 13.10, the distance from every point in D to p is less than $\varepsilon f(p)$. ◻

13.5 Global properties of restricted Voronoi faces

The Small Intersection Theorem does not rule out the possibility that a restricted Voronoi face may have misbehaved components that do not adjoin a restricted Voronoi vertex, as illustrated at center in Figure 13.13. It only guarantees that, if certain local constraints are imposed, a restricted Voronoi face has at most one connected component that adjoins a restricted Voronoi vertex, ruling out the configuration at left in Figure 13.13; and that component is a topological ball. The forthcoming Voronoi Intersection Theorem shows that if Σ_p is ε-small for *every* sample point p and at least one edge in $\operatorname{Vor} S$ intersects the surface Σ, then every restricted Voronoi face is a topological ball, as illustrated at right in Figure 13.13. Moreover, it guarantees that the topological ball property holds.

Proposition 13.13. *Let S be a finite sample of a connected, smooth surface Σ. If some edge in* $\operatorname{Vor} S$ *intersects Σ and there is an $\varepsilon \le 0.09$ such that Σ_p is ε-small for every $p \in S$, then*

$V_p \cap \Sigma = \Sigma_p$ for every $p \in S$.

Proof. Assume for the sake of contradiction that there is a sample point $q \in S$ such that $V_q \cap \Sigma \neq \Sigma_q$. It follows from the Small Intersection Theorem that some connected component σ_q of $V_q \cap \Sigma$ does not intersect any Voronoi edge.

By assumption, there exists a sample point s such that some edge of V_s intersects Σ. By the Voronoi Cell Lemma (Lemma 13.12), Σ_s is a topological disk. Consider a path in Σ connecting a point in σ_q to a point in Σ_s. Let $q = p_0, p_1, ..., p_k = s$ be the sequence of sample points in S whose Voronoi cells are visited along this path. Let σ_{p_i} be the connected component of $V_{p_i} \cap \Sigma$ visited by the path when it visits V_{p_i} in the sequence. There must be two consecutive points p_i and p_{i+1} in this sequence such that σ_{p_i} does not intersect any Voronoi edge and $\sigma_{p_{i+1}}$ intersects a Voronoi edge, because σ_{p_0} does not intersect any Voronoi edge and $\sigma_{p_k} = \Sigma_s$ does.

The boundaries of σ_{p_i} and $\sigma_{p_{i+1}}$ intersect. Because the former boundary intersects no Voronoi edge, the intersection must be one or more complete boundary loops. By the Voronoi Cell Lemma (Lemma 13.12), $\sigma_{p_{i+1}}$ is the topological disk $\Sigma_{p_{i+1}}$; hence, $\sigma_{p_{i+1}}$ has only one boundary loop. But this loop intersects a Voronoi edge, a contradiction. □

Theorem 13.14 (Voronoi Intersection Theorem)**.** *Let S be a finite sample of a connected, smooth surface Σ. If some edge in* Vor S *intersects Σ and there is an $\varepsilon \leq 0.09$ such that Σ_p is ε-small for every $p \in S$, then the following properties hold for every $p \in S$.*

(i) $V_p|_\Sigma = \Sigma_p$.

(ii) *Σ_p is a topological disk and the distance from p to every point in Σ_p is less than $\varepsilon f(p)$.*

(iii) *Every edge of* Vor S *that intersects Σ does so transversally at a single point.*

(iv) *Every facet of* Vor S *that intersects Σ does so transversally in a topological interval.*

(v) *The point set S is an $\varepsilon/(1-\varepsilon)$-sample of Σ.*

Proof. Assertions (i)–(iv) follow from the Small Intersection Theorem (Theorem 13.6) and Proposition 13.13. For assertion (v), let x be a point in Σ. It follows from (i) that $x \in \Sigma_p$ for some $p \in S$. Assertion (ii) and the Feature Translation Lemma (Lemma 12.2) give $d(p, x) < \varepsilon f(p) \leq \frac{\varepsilon}{1-\varepsilon} f(x)$. Hence, S is an $\varepsilon/(1-\varepsilon)$-sample of Σ. □

The Voronoi Intersection Theorem offers stronger guarantees than the Small Intersection Theorem (Theorem 13.6). First, every restricted Voronoi cell $V_p|_\Sigma$ is a topological disk. Second, a facet of Vor S cannot intersect Σ in a loop or degenerate curve. Third, the topological ball property holds. Fourth, S is guaranteed to be a dense sample of Σ.

The preconditions of the Voronoi Intersection Theorem imply that S is dense, being an $\varepsilon/(1-\varepsilon)$-sample. The implication can be reversed, thereby showing that a sufficiently dense sample guarantees the same good consequences.

Theorem 13.15 (Dense Sample Theorem)**.** *Let S be an ε-sample of Σ for some $\varepsilon \leq 0.08$. Then, for every $p \in S$, Σ_p is nonempty and 0.09-small. Moreover, the consequences of the Voronoi Intersection Theorem (Theorem 13.14) hold.*

Proof. Suppose for the sake of contradiction that some Σ_p is empty. Then the loops in $(\mathrm{Bd}\, V_p) \cap \Sigma$ reside in the relative interiors of the facets of V_p. Let C be such a loop and let g be the facet of V_p that includes C. Consider a line $\ell_x \subset \mathrm{aff}\, g$ that is perpendicular to C at a point $x \in C$. The line ℓ_x intersects C at another point y because C is a loop in a plane. As S is an ε-sample and p is a nearest sample point to x, $d(p, x) \leq \varepsilon f(x) \leq \frac{\varepsilon}{1-\varepsilon} f(p)$ by the Feature Translation Lemma (Lemma 12.2). Similarly, $d(p, y) \leq \frac{\varepsilon}{1-\varepsilon} f(p)$. As $\varepsilon \leq 0.08$, $d(p, x) < 0.09 f(p)$ and $d(p, y) < 0.09 f(p)$. The line ℓ_x contains the projection of $\mathbf{n}_x$ onto g because it is perpendicular to C at x. By Proposition 13.3(ii), we have $\angle_a(xy, \mathbf{n}_x) = \angle_a(\ell_x, \mathbf{n}_x) \leq \alpha(0.09) + \arcsin 0.09$, which is less than $\alpha(0.09) + \beta(0.09)$. But then Proposition 13.5 implies that one of $d(p, x)$ or $d(p, y)$ is greater than $0.09 f(p)$, a contradiction.

Because Σ_p is nonempty for every $p \in S$, it follows that Σ intersects a Voronoi edge. For any $p \in S$, let x be a restricted Voronoi vertex of Σ_p. As S is an ε-sample and p is a sample point nearest x, $d(p, x) \leq \frac{\varepsilon}{1-\varepsilon} f(p)$. It follows that Σ_p is $\varepsilon/(1-\varepsilon)$-small and thus 0.09-small for $\varepsilon \leq 0.08$.

The assumptions of the Voronoi Intersection Theorem are satisfied and its consequences follow. □

13.6 The fidelity of the restricted Delaunay triangulation

The meshing algorithms described in subsequent chapters generate the restricted Delaunay triangulation $\mathrm{Del}|_\Sigma S$ or a tetrahedral mesh bounded by $\mathrm{Del}|_\Sigma S$. Here we use the Voronoi Intersection Theorem (Theorem 13.14) to show that the underlying space of $\mathrm{Del}|_\Sigma S$ is topologically equivalent to Σ and that the two manifolds are geometrically similar. For a precise statement of the nature of the geometric similarity, see the Surface Discretization Theorem (Theorem 13.22) at the end of this chapter.

Observe that the topological ball property is one consequence of the Voronoi Intersection Theorem (Theorem 13.14). We put it together with the Topological Ball Theorem.

Theorem 13.16 (Homeomorphism Theorem)**.** *Let S be a sample of a connected, smooth surface $\Sigma \subset \mathbb{R}^3$. If some edge in $\mathrm{Vor}\, S$ intersects Σ and there is an $\varepsilon \leq 0.09$ such that for every $p \in S$, Σ_p is ε-small, then the underlying space of $\mathrm{Del}|_\Sigma S$ is homeomorphic to Σ.*

Thus $\mathrm{Del}|_\Sigma S$ is a triangulation of Σ by Definition 12.6.

13.6.1 The nearest point map is a homeomorphism

The Homeomorphism Theorem is purely topological; it establishes that, under the right conditions, $\mathrm{Del}|_\Sigma S$ is a triangulation of Σ, but it does not say that they are geometrically similar. Most of the remainder of this chapter is devoted to showing that $|\mathrm{Del}|_\Sigma S|$ and Σ can be continuously deformed to one another by small perturbations that maintain a homeomorphism between them; thus, they are related by an isotopy. Let M be the medial axis of Σ. To construct the homeomorphism, we recall from Section 12.6 the function that maps every point $z \in \mathbb{R}^3 \setminus M$ to the unique point $\tilde{z}$ nearest z on Σ. We call this function the *nearest point map*.

Definition 13.5 (nearest point map)**.** Let $\mathcal{T}$ be a triangulation of a 2-manifold $\Sigma \subset \mathbb{R}^3$. The *nearest point map* on $\mathcal{T}$ maps every point $z \in |\mathcal{T}|$ to the nearest point $\tilde{z}$ on Σ.

For the nearest point map to be defined, $|\mathcal{T}|$ must be disjoint from the medial axis M. If that restriction is satisfied, the nearest point map is continuous. For the nearest point map to be bijective, and therefore a homeomorphism, the triangulation must satisfy additional conditions, which we encapsulate in the notion of *ε-dense triangulations*.

Definition 13.6 (ε-dense)**.** A triangulation $\mathcal{T}$ of a 2-manifold $\Sigma \subset \mathbb{R}^3$ is *ε-dense* if

(i) the vertices in $\mathcal{T}$ lie on Σ,

(ii) for every triangle τ in $\mathcal{T}$ and every vertex p of τ, the circumradius of τ is at most $\varepsilon f(p)$, and

(iii) $|\mathcal{T}|$ can be oriented such that for every triangle τ in $\mathcal{T}$ and every vertex p of τ, the angle between $\mathbf{n}_p$ and the vector $\mathbf{n}_\tau$ normal to τ is at most $\pi/2$.

One implication of this definition is that for sufficiently small ε, an ε-dense triangulation of Σ does not intersect the medial axis.

Proposition 13.17. *For $\varepsilon < 0.5$, an ε-dense triangulation $\mathcal{T}$ of Σ does not intersect the medial axis of Σ.*

Proof. Let τ be an arbitrary triangle in $\mathcal{T}$. Let p be a vertex of τ. Because $\mathcal{T}$ is ε-dense, τ lies in $B(p, 2\varepsilon f(p))$, which is included in the interior of $B(p, f(p))$ as $\varepsilon < 0.5$. By the definition of LFS, the medial axis of Σ intersects the boundary of $B(p, f(p))$ but not its interior. It follows that τ is disjoint from the medial axis of Σ. □

Another implication is that for $\varepsilon \leq 0.09$, the nearest point map is bijective and so it is a homeomorphism. We will also use it to define an isotopy relating $|\mathrm{Del}|_\Sigma S|$ to Σ.

Proposition 13.18. *If there is an $\varepsilon \leq 0.09$ such that $\mathcal{T}$ is an ε-dense triangulation of Σ, then the nearest point map on $\mathcal{T}$ is bijective.*

Proof. For every point $x \in \Sigma$, let ℓ_x be the line segment connecting the centers of the two medial balls at x. Observe that ℓ_x is orthogonal to Σ at x. If there is only one medial ball at x, let ℓ_x be the ray originating at the center of the medial ball and passing through x.

Let $\nu : |\mathcal{T}| \to \Sigma$ be the nearest point map of $\mathcal{T}$. Let z be a point in $\ell_x \cap |\mathcal{T}|$. Observe that the ball $B(z, d(x, z))$ is included in the medial ball at x containing z. Recall that Σ does not intersect the interiors of the medial balls, so x is the point nearest z on Σ. Hence, $\tilde{z} = x$ for every point $z \in \ell_x \cap |\mathcal{T}|$.

Now we show that ν is injective: for every point $x \in \Sigma$, ℓ_x intersects $|\mathcal{T}|$ in at most one point. Assume to the contrary that for some $x \in \Sigma$, ℓ_x intersects $|\mathcal{T}|$ at two points y_1 and y_2.

We find a triangle τ intersecting ℓ_x such that $\mathbf{n}_x$ makes an angle at least $\pi/2$ with the normal $\mathbf{n}_\tau$ to τ. Such a triangle always exists. This claim is trivial if ℓ_x is parallel to a triangle that it intersects. Suppose that it is not. Then y_1 and y_2 lie in distinct triangles not parallel to ℓ_x. We can thus choose y_1 and y_2 to be consecutive intersection points in $\ell_x \cap |\mathcal{T}|$. Without loss of generality, we assume that ℓ_x enters the volume bounded by $|\mathcal{T}|$ at y_1 and leaves this volume at y_2. Then, $\mathbf{n}_x$ makes an angle more than $\pi/2$ with the normal to one of the two triangles containing y_1 and y_2.

Let y be an intersection point in $\ell_x \cap \tau$. Let v be the vertex of τ having the largest angle in τ. Because $\mathcal{T}$ is ε-dense, the circumradius of τ is at most $\varepsilon f(v)$. By the Triangle Normal Lemma (Lemma 12.14), $\angle(\mathbf{n}_v, \mathbf{n}_\tau) < \arcsin \varepsilon + \arcsin\left(\frac{2}{\sqrt{3}} \sin(2 \arcsin \varepsilon)\right)$. Therefore,

$$\begin{aligned} \angle(\mathbf{n}_v, \mathbf{n}_x) &\geq \angle(\mathbf{n}_x, \mathbf{n}_\tau) - \angle(\mathbf{n}_v, \mathbf{n}_\tau) \\ &\geq \frac{\pi}{2} - \arcsin \varepsilon - \arcsin\left(\frac{2}{\sqrt{3}} \sin(2 \arcsin \varepsilon)\right). \end{aligned} \tag{13.1}$$

On the other hand, since the circumradius of τ is at most $\varepsilon f(v)$, $d(v, y) \leq 2\varepsilon f(v)$. We have argued that $\tilde{y} = x$. So $d(x, y) \leq d(v, y)$. This gives $d(v, x) \leq d(x, y) + d(v, y) \leq 4\varepsilon f(v)$ and so $\angle(\mathbf{n}_v, \mathbf{n}_x) \leq \alpha(4\varepsilon)$ by the Normal Variation Theorem (Theorem 12.8). We reach a contradiction because $\alpha(4\varepsilon)$ is less than the lower bound (13.1). We conclude that, for any point $x \in \Sigma$, there is at most one point in $\ell_x \cap |\mathcal{T}|$.

It remains to show that ν is surjective: for every point $x \in \Sigma$, ℓ_x intersects $|\mathcal{T}|$. Assume to the contrary that for some point $x_0 \in \Sigma$, ℓ_{x_0} does not intersect $|\mathcal{T}|$. As $|\mathcal{T}|$ is disjoint from the medial axis of Σ, we can pick any point $z' \in |\mathcal{T}|$, and there is a unique point $x_1 \in \Sigma$ closest to z', implying that ℓ_{x_1} intersects $|\mathcal{T}|$ at z'. Move a point x continuously from x_0 to x_1 on Σ, and stop when ℓ_x intersects $|\mathcal{T}|$ for the first time. When we stop, the segment ℓ_x is tangent to $|\mathcal{T}|$ at a point y. Let τ be a triangle in $\mathcal{T}$ containing y. The point y cannot lie in the interior of τ because ℓ_x would then intersect τ in more than one point, violating the injectivity of ν. So y lies in a boundary edge e of τ. We can move to a point $x' \in \Sigma$ arbitrarily close to x such that $\ell_{x'}$ intersects $|\mathcal{T}|$ in more than one point because each edge in $\mathcal{T}$ lies in two triangles. We reach a contradiction to the injectivity of ν. □

13.6.2 Proximity and isotopy

The geometric similarity between two point sets is often measured in terms of Hausdorff distances. The *Hausdorff distance* between two point sets $A, B \subseteq \mathbb{R}^d$ is

$$d_H(A, B) = \max\{\max_{y \in B} d(y, A), \max_{x \in A} d(x, B)\};$$

that is, we find the point that is farthest from the nearest point in the other object. Observe that if the maxima are replaced by minima, we have $d(A, B)$ instead.

We will show that the nearest point map sends a point $z \in |\mathcal{T}|$ over a distance that is tiny compared to the local feature size at $\tilde{z}$, the point nearest to z on Σ. It follows that the Hausdorff distance between $|\mathcal{T}|$ and Σ is small in terms of the maximum local feature size.

Proposition 13.19. *Let $\mathcal{T}$ be an ε-dense triangulation of Σ for some $\varepsilon < 0.09$. Then, $d(z, \tilde{z}) < 15\varepsilon^2 f(\tilde{z})$ for every point $z \in |\mathcal{T}|$.*

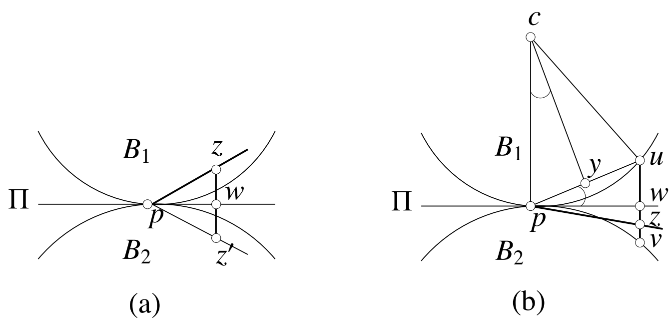


Figure 13.14: Proposition 13.19. (a) Case 1: z is in B_1. (b) Case 2: z is outside B_1 and B_2.

Proof. Let z be a point in $|\mathcal{T}|$. Let τ be a triangle in $\mathcal{T}$ that contains z. Let p be the vertex of τ nearest to z. Therefore, $d(p,z) \leq \varepsilon f(p)$. By the Triangle Normal Lemma (Lemma 12.14) and the Normal Variation Theorem (Theorem 12.8),

$$\begin{aligned}\angle_a(\mathbf{n}_\tau, \mathbf{n}_p) &\leq \frac{2\varepsilon}{1-2\varepsilon} + \arcsin\varepsilon + \arcsin\left(\frac{2}{\sqrt{3}}\sin(2\arcsin\varepsilon)\right) \\ &< 6\varepsilon \ \text{ for } \varepsilon < 0.09.\end{aligned}$$

This implies that for any point $r \in \tau$, the segment pr makes an angle less than 6ε with the plane tangent to Σ at p.

Let B_1 and B_2 be two balls with radii $f(p)$, tangent to Σ at p, and lying on opposite sides of Σ. The medial balls at p are also tangent to Σ at p and their radii are $f(p)$ or larger. So B_1 and B_2 are included in these two medial balls, and Σ does not intersect the interiors of B_1 and B_2.

There are two cases in bounding the distance $d(z,\tilde{z})$, depending on whether $z \in B_1 \cup B_2$.

- Case 1: z lies in $B_1 \cup B_2$. Assume without loss of generality that z lies in B_1. Let Π be the tangent plane at p. Let z' be the reflection of z with respect to Π. By symmetry, the point z' lies in B_2. Refer to Figure 13.14(a). As B_1 and B_2 lie on opposite sides of Σ, the segment zz' intersects Σ. Let x be an intersection point in $zz' \cap \Sigma$. Clearly, $d(z,\tilde{z}) \leq d(z,x) \leq d(z,z')$. By construction, zz' also intersects the tangent plane Π orthogonally at a point w. Consider the right-angled triangle zpw. We have

$$\begin{aligned}d(z,w) &= d(p,z)\sin\angle zpw \\ &\leq \varepsilon f(p)\sin(6\varepsilon) \\ &< 5.72\varepsilon^2 f(p). \qquad (13.2)\end{aligned}$$

  It follows that

$$d(z,\tilde{z}) \leq d(z,z') = 2d(z,w) < 11.5\varepsilon^2 f(p).$$

- Case 2: z lies outside $B_1 \cup B_2$. Extend a line segment through z perpendicular to Π until the extension stops at points u and v on the boundaries of B_1 and B_2, respectively. Again, uv intersects Σ, which is sandwiched between B_1 and B_2. Refer to

Figure 13.14(b). Either the segment zu or zv intersects Σ. It follows that $d(z, \tilde{z})$ is at most $\max\{d(z, u), d(z, v)\}$. Assume without loss of generality that $d(z, u) \geq d(z, v)$, so u and z lie on opposite sides of the plane Π.

Let $w = zu \cap \Pi$. The same analysis that obtains (13.2) applies here, so

$$d(z, w) < 5.72\varepsilon^2 f(p).$$

If we can bound $d(u, w)$, then a bound on $d(z, u)$ follows since $d(z, u) = d(z, w) + d(u, w)$. Consider the triangle puw in Figure 13.14(b). Then

$$d(p, u) \cos \angle upw \;=\; d(p, w) \;\leq\; d(p, z) \leq \varepsilon f(p).$$

At the same time,

$$d(p, u) \;=\; 2f(p) \sin \angle pcy \;=\; 2f(p) \sin \angle upw.$$

Thus,

$$\begin{array}{rrcl} & 2f(p) \sin \angle upw \cos \angle upw & \leq & \varepsilon f(p) \\ \Rightarrow & \angle upw & \leq & (\arcsin \varepsilon)/2. \end{array}$$

It follows that

$$d(u, w) \;\leq\; d(p, w) \tan \angle upw \;\leq\; \varepsilon f(p) \tan\left(\frac{1}{2} \arcsin \varepsilon\right) \;\leq\; \varepsilon^2 f(p).$$

Hence, $d(z, u) \leq d(z, w) + d(u, w) < 6.72\varepsilon^2 f(p)$.

Cases 1 and 2 together yield $d(z, \tilde{z}) \leq 11.5\varepsilon^2 f(p)$. By the triangle inequality and the fact that $\mathcal{T}$ is ε-dense, $d(p, \tilde{z}) \leq d(p, z) + d(z, \tilde{z}) \leq 2.1\varepsilon f(p)$. The Feature Translation Lemma (Lemma 12.2) implies that $11.5\varepsilon^2 f(p) \leq 11.5\varepsilon^2 f(\tilde{z})/(1 - 2.1\varepsilon) < 15\varepsilon^2 f(\tilde{z})$. □

Next, we use the nearest point map of an ε-dense triangulation $\mathcal{T}$ to construct an isotopy relating $|\mathcal{T}|$ and Σ. Each point in $|\mathcal{T}|$ is very close to its image in Σ under the isotopy.

Proposition 13.20. *If there is an $\varepsilon < 0.09$ such that $\mathcal{T}$ is an ε-dense triangulation of Σ, then the nearest point map of $\mathcal{T}$ induces an ambient isotopy that moves each point $x \in \Sigma$ by a distance less than $15\varepsilon^2 f(x)$, and each point $z \in |\mathcal{T}|$ by a distance less than $15\varepsilon^2 f(\tilde{z})$.*

Proof. We define a map $\xi : \mathbb{R}^3 \times [0, 1] \to \mathbb{R}^3$ such that $\xi(|\mathcal{T}|, 0) = |\mathcal{T}|$ and $\xi(|\mathcal{T}|, 1) = \Sigma$ and $\xi(\cdot, t)$ is a continuous and bijective map for all $t \in [0, 1]$. Consider the following tubular neighborhood of Σ.

$$N_\Sigma = \{z \in \mathbb{R}^3 : d(z, \Sigma) \leq 15\varepsilon^2 f(\tilde{z})\}.$$

Proposition 13.19 implies that $|\mathcal{T}| \subset N_\Sigma$. We define an ambient isotopy ξ as follows. For $z \in \mathbb{R}^3 \setminus N_\Sigma$, ξ is the identity on z for all $t \in [0, 1]$; that is, $\xi(z, t) = z$. For $z \in N_\Sigma$, we use a map that moves points on $|\mathcal{T}|$ toward their closest points on Σ. Consider the line segment s that is normal to Σ at $\tilde{z}$, thus passing through z, and has endpoints s_i and s_o on the two boundary surfaces of N_Σ. Because $15\varepsilon^2 < 1$, the tubular neighborhood N_Σ is disjoint from the medial axis of Σ; thus, s intersects Σ only at $\tilde{z}$. Likewise, by Proposition 13.18, s intersects $|\mathcal{T}|$ only

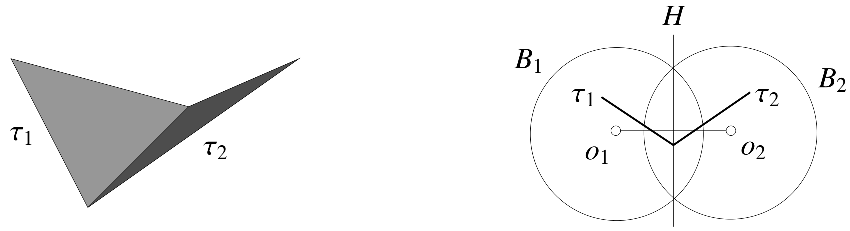


Figure 13.15: Two triangles that share an edge.

at a single point $\dot{z}$. Let $z_t = (1-t)\dot{z} + t\tilde{z}$ be a point that moves linearly from $\dot{z} \in |\mathcal{T}|$ at time zero to $\tilde{z} \in \Sigma$ at time 1. Let $\xi(\cdot, t)$ linearly map the segments $s_i\dot{z}$ to $s_i z_t$ and $s_o\dot{z}$ to $s_o z_t$. That is,

$$\xi(z,t) = \begin{cases} s_i + \dfrac{d(z_t, s_i)}{d(\dot{z}, s_i)}(z - s_i) & \text{if } z \in s_i\dot{z}, \\[2ex] s_o + \dfrac{d(z_t, s_o)}{d(\dot{z}, s_o)}(z - s_o) & \text{if } z \in s_o\dot{z}. \end{cases}$$

The map ξ is continuous and bijective for all $z \in \mathbb{R}^3$ and $t \in [0,1]$, so it is an ambient isotopy, and $\xi(|\mathcal{T}|, 0) = |\mathcal{T}|$ and $\xi(|\mathcal{T}|, 1) = \Sigma$. ◻

13.6.3 Fidelity and dihedral angles of the discretized surface

We wish to prove that the restricted Delaunay triangulation is ε-dense and apply Proposition 13.20 to conclude that $\mathrm{Del}|_\Sigma S$ approximates Σ. It requires some work to show that $\mathrm{Del}|_\Sigma S$ satisfies the orientation property (iii) in Definition 13.6. The first step is to prove that the dihedral angles between adjoining triangles are obtuse.

Proposition 13.21. *Let τ_1 and τ_2 be two triangles in $\mathrm{Del}|_\Sigma S$ that share an edge. Suppose that there is an $\varepsilon \le 0.09$ such that for every vertex p of each $\tau \in \{\tau_1, \tau_2\}$, there is an empty circumball of τ with radius at most $\varepsilon f(p)$ centered at a point on Σ. Then τ_1 and τ_2 meet at a dihedral angle greater than $\pi/2$.*

Proof. For $i \in \{1,2\}$, let B_i be the smallest empty circumball of τ_i whose center o_i lies on Σ. The boundaries of B_1 and B_2 intersect in a circle C. Let $H = \mathrm{aff}\, C$. The plane H contains the common edge of τ_1 and τ_2, as illustrated in Figure 13.15. The triangles τ_1 and τ_2 must lie on opposite sides of H; otherwise, either the interior of B_1 would contain a vertex of τ_2 or the interior of B_2 would contain a vertex of τ_1.

We have $d(o_1, o_2) \le d(p, o_1) + d(p, o_2) \le 2\varepsilon f(p)$, which implies that $d(o_1, o_2) \le \frac{2\varepsilon}{1-\varepsilon} f(o_1)$ by the Three Points Lemma (Lemma 12.3). Hence the segment $o_1 o_2$ meets $\mathbf{n}_{o_1}$ at an angle of at least $\pi/2 - \arcsin \frac{\varepsilon}{1-\varepsilon}$ by the Edge Normal Lemma (Lemma 12.12). In turn, $\angle_a(\mathbf{n}_{o_1}, \mathbf{n}_p) \le \varepsilon/(1-\varepsilon)$ by the Normal Variation Theorem (Theorem 12.8). It follows that the angle between H and $\mathbf{n}_p$ is at most $\varepsilon/(1-\varepsilon) + \arcsin \varepsilon/(1-\varepsilon)$, which is less than 0.2 radians for $\varepsilon \le 0.09$.

As $\varepsilon \le 0.09$, the Triangle Normal Lemma (Lemma 12.14) and the Normal Variation Theorem (Theorem 12.8) imply that the normal of τ_1 differs from $\mathbf{n}_p$ by at most

$$\arcsin 0.09 + \arcsin\left(\frac{2}{\sqrt{3}}\sin(2\arcsin 0.09)\right) + \frac{0.18}{1-0.18} < 0.52.$$

The angle between the normal of τ_2 and $\mathbf{n}_p$ is also less than 0.52 radians.

Therefore, τ_1 and τ_2 lie on opposite sides of H and the angle between H and either triangle is greater than $\pi/2 - (0.52 + 0.2) > 0.85$. So, the dihedral angle between τ_1 and τ_2 is greater than $1.7 > \pi/2$ radians. $\square$

We come to the climax of the chapter, the Surface Discretization Theorem, which states that the underlying space of $\mathrm{Del}|_\Sigma S$ is related by isotopy and geometrically close to Σ. It also states several other geometric properties of $\mathrm{Del}|_\Sigma S$, including an upper bound on the circumradii of the triangles, an upper bound on the deviation between the surface normal at a sample point and the normal of any triangle having it as a vertex, and a lower bound on the dihedral angle between two adjoining triangles.

Theorem 13.22 (Surface Discretization Theorem)**.** *Let S be a sample of a connected, smooth surface $\Sigma \subset \mathbb{R}^3$. If some edge in* $\mathrm{Vor}\, S$ *intersects Σ and there is an $\varepsilon \le 0.09$ such that Σ_p is ε-small for every $p \in S$, then* $\mathrm{Del}|_\Sigma S$ *is a triangulation of Σ with the following properties.*

(i) *For every triangle τ in* $\mathrm{Del}|_\Sigma S$ *and every vertex p of τ, the circumradius of τ is at most $\varepsilon f(p)$.*

(ii) $|\mathrm{Del}|_\Sigma S|$ *can be oriented so that for every triangle $\tau \in \mathrm{Del}|_\Sigma S$ and every vertex p of τ, the angle between $\mathbf{n}_p$ and the oriented normal $\mathbf{n}_\tau$ of τ is less than 7ε.*

(iii) *Any two triangles in* $\mathrm{Del}|_\Sigma S$ *that share an edge meet at a dihedral angle greater than $\pi - 14\varepsilon$.*

(iv) *The nearest point map $\nu : |\mathrm{Del}|_\Sigma S| \to \Sigma, z \mapsto \tilde{z}$ is a homeomorphism that induces an isotopy relating $|\mathrm{Del}|_\Sigma S|$ to Σ.*

(v) *For every point z in $|\mathrm{Del}|_\Sigma S|$, $d(z, \tilde{z}) < 15\varepsilon^2 f(\tilde{z})$.*

Proof. We argue that $\mathrm{Del}|_\Sigma S$ is a ε-dense triangulation of Σ. The correctness of (i)–(v) is established along the way.

By the Homeomorphism Theorem (Theorem 13.16), the underlying space of $\mathrm{Del}|_\Sigma S$ is homeomorphic to Σ and $\mathrm{Del}|_\Sigma S$ is a triangulation of Σ. Let τ be a triangle in $\mathrm{Del}|_\Sigma S$ and let p be a vertex of τ. Because Σ_p is ε-small for every $p \in S$, the dual edge of τ in $\mathrm{Del}|_\Sigma S$ intersects Σ at a point within a distance of $\varepsilon f(p)$ from p. Therefore, τ has an empty circumball with radius at most $\varepsilon f(p)$ whose center lies on Σ. As this is true for every vertex of every triangle in $\mathrm{Del}|_\Sigma S$, Proposition 13.21 applies.

By Proposition 13.2, the circumradius of τ is at most $\varepsilon f(p)$, proving (i). By Proposition 13.4, $\angle_a(\mathbf{n}_\tau, \mathbf{n}_p) \le \beta(\varepsilon)$, which is less than 7ε for $\varepsilon \le 0.09$. Furthermore, as no dihedral angle is less than $\pi/2$ by Proposition 13.21, one can orient the triangles in $\mathrm{Del}|_\Sigma S$ consistently so that for any $p \in S$ and for any triangle τ adjoining p, the angle between $\mathbf{n}_p$ and the oriented normal of τ is less than 7ε, proving (ii). The correctness of (iii) follows by comparing the oriented normals of two adjoining triangles with the surface normal at a common vertex.

We conclude that $\mathrm{Del}|_\Sigma S$ is an ε-dense triangulation of Σ. Thus Proposition 13.20 gives (iv) and (v). □

13.7 Notes and exercises

The earliest appearance of the restricted Delaunay triangulation of a surface is in a paper by Chew [61] on Delaunay surface meshing. Given a sufficiently fine triangulation of a surface, Chew presents methods to make the triangulation "Delaunay" through edge flips and to refine the triangulation so it satisfies quality guarantees. He did not recognize that his triangulations are usually a subcomplex of the Delaunay tetrahedralization. Restricted Delaunay triangulations were formally introduced by Edelsbrunner and Shah [92], who also introduced the topological ball property and proved the Topological Ball Theorem (Theorem 13.1).

The earliest applications of restricted Delaunay triangulations were in algorithms for reconstructing curves and surfaces from point clouds. These algorithms try to recover the shape and topology of an object from a dense sample of points collected from the object's surface by a laser range scanner or stereo photography. Amenta and Bern [3] show that for a sufficiently dense sample, the Voronoi cells are elongated along the directions normal to the surface, and that the Voronoi diagram satisfies the topological ball property. Hence, the restricted Delaunay triangulation is homeomorphic to the surface.

Cheng, Dey, Edelsbrunner, and Sullivan [47] adapt these ideas to develop a theory of surface sampling for generating an ε-sample whose restricted Delaunay triangulation is homeomorphic to a surface, which they use to triangulate specialized surfaces for molecular modeling. This paper includes early versions of the Voronoi Edge, Voronoi Facet, and Voronoi Cell Lemmas that require an ε-sample, which is a global property. The stronger, local versions of the Voronoi Edge and Voronoi Facet Lemmas presented here are adapted from Cheng, Dey, Ramos, and Ray [53], who use them to develop a surface meshing algorithm described in the next chapter. The versions of the Voronoi Cell Lemma and the Small Intersection Theorem (Theorem 13.6) in this chapter appear for the first time here.

Boissonnat and Oudot [29, 30] independently developed a similar theory of surface sampling and applied it to the more general problem of guaranteed-quality Delaunay refinement meshing of smooth surfaces. The notion of an ε-small Σ_p, used heavily here, is related to Boissonnat and Oudot's notion of a loose ε-sample [30]. They prove the Surface Discretization Theorem (Theorem 13.22) not by using the topological ball property, but by showing that restricted Delaunay triangulations of loose ε-samples satisfy the preconditions for homeomorphic meshing outlined by Amenta, Choi, Dey, and Leekha [6]: a simplicial complex $\mathcal{T}$ has $|\mathcal{T}|$ homeomorphic to Σ if it satisfies the following four conditions. (i) All the vertices in $\mathcal{T}$ lie on Σ. (ii) $|\mathcal{T}|$ is a 2-manifold. (iii) All the triangles in $\mathcal{T}$ have small circumradii compared to their local feature sizes. (iv) The triangle normals approximate closely the surface normals at their vertices. Boissonnat and Oudot [29, 30] extend the homeomorphism to an isotopy with a nearest point map similar to that of Definition 13.5. Here we prove the Discretization Theorem more directly by showing that the topological ball property holds for loose ε-samples, then extending the homeomorphism to isotopy.

Exercises

1. Let Σ be a 2-manifold without boundary in $\mathbb{R}^3$. Let $S \subset \Sigma$ be a finite point sample. Let τ_1 and τ_2 be two tetrahedra in $\operatorname{Del} S$ that share a triangular face $\sigma \subset \Sigma$. In other words, the surface Σ just happens to have a flat spot that includes the shared face.

   Give a specific example that shows why σ might nonetheless not appear in the restricted Delaunay triangulation $\operatorname{Del}|_\Sigma S$.

2. [3] Let S be an ε-sample of a smooth surface Σ without boundary. For a sample point $p \in S$, the Voronoi vertex $v \in V_p$ farthest from p is called the *pole* of p and the vector $\mathbf{v}_p = v - p$ is called the *pole vector* of p. Prove that if $\varepsilon < 1$, then $\angle_a(\mathbf{n}_p, \mathbf{v}_p) \leq 2 \arcsin \frac{\varepsilon}{1-\varepsilon}$.

3. Let S be an ε-sample of a smooth surface Σ without boundary for some $\varepsilon \leq 0.1$. Prove that the intersection of Σ and a Voronoi facet in $\operatorname{Vor} S$ is either empty or a topological interval.

4. Let S be an ε-sample of a smooth surface Σ without boundary for some $\varepsilon \leq 0.1$. Prove that for any $p \in S$, the restricted Voronoi cell $V_p \cap \Sigma$ is a topological disk.

5. Construct an example where Σ_p is 0.09-small for exactly one sample point p in S.

6. Show that the TBP is not necessary for the underlying space of $\operatorname{Del}|_\Sigma S$ to be homeomorphic to Σ.

7. Let S be a sample of a smooth, compact manifold Σ without boundary in $\mathbb{R}^3$.

   (a) Show an example where Σ is a 1-manifold and the TBP holds, but there exists no isotopy that continuously deforms $|\operatorname{Del}|_\Sigma S|$ to Σ.

   (b) If Σ is a surface, prove or disprove that if the TBP holds, there is an isotopy relating $|\operatorname{Del}|_\Sigma S|$ to Σ.

8. [6] Let S be a sample of a smooth surface Σ without boundary. Let $\mathcal{T}$ be a subcomplex of $\operatorname{Del} S$ such that $|\mathcal{T}|$ is a 2-manifold and the Voronoi edge dual to each triangle $\tau \in \mathcal{T}$ intersects Σ, and all the intersection points are within a distance of $0.09 f(p)$ from a vertex p of τ. Prove that $|\mathcal{T}|$ is homeomorphic to Σ.

9. [93] Let Σ be a smooth surface without boundary. Let $\lambda : \Sigma \to \mathbb{R}$ be a 1-Lipschitz function that specifies an upper bound on the spacing of points in a sample $S \subset \Sigma$: for every point $x \in \Sigma$, $d(x, S) \leq \lambda(x)$. Prove that if $\lambda(x) \leq f(x)/5$ for every $x \in \Sigma$, then S contains $\Omega\left(\int_\Sigma dx/\lambda(x)^2\right)$ sample points. For example, every ε-sample of Σ for any $\varepsilon \leq 1/5$ has $\Omega\left(\varepsilon^{-2} \int_\Sigma dx/f(x)^2\right)$ sample points.