Natural Language Processing

Classification II

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Classification

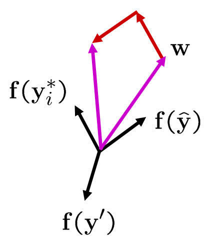
Linear Models: Perceptron

- The perceptron algorithm
  - Iteratively processes the training set, reacting to training errors
  - Can be thought of as trying to drive down training error
- The (online) perceptron algorithm:
  - Start with zero weights w
  - Visit training instances one by one
    - Try to classify

$$\hat{\mathbf{y}} = \arg\max_{\mathbf{y} \in \mathcal{Y}(\mathbf{x})} \mathbf{w}^\top \mathbf{f}(\mathbf{y})$$

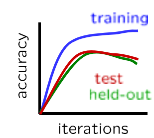
    - If correct, no change!
    - If wrong: adjust weights

$$\mathbf{w} \leftarrow \mathbf{w} + \mathbf{f}(\mathbf{y}_i^*)$$
$$\mathbf{w} \leftarrow \mathbf{w} - \mathbf{f}(\hat{\mathbf{y}})$$

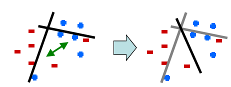


Issues with Perceptrons

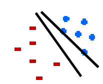
- Overtraining: test / held-out accuracy usually rises, then falls
  - Overtraining isn't the typically discussed source of overfitting, but it can be important



- Regularization: if the data isn't separable, weights often thrash around
  - Averaging weight vectors over time can help (averaged perceptron)
  - [Freund & Schapire 99, Collins 02]



- Mediocre generalization: finds a "barely" separating solution

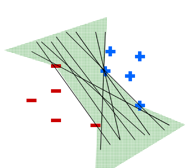


Problems with Perceptrons

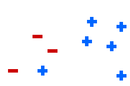
- Perceptron "goal": separate the training data

$$\forall i, \forall \mathbf{y} \neq \mathbf{y}^i \quad \mathbf{w}^\top \mathbf{f}_i(\mathbf{y}^i) \geq \mathbf{w}^\top \mathbf{f}_i(\mathbf{y})$$

1. This may be an entire feasible space



2. Or it may be impossible



Margin

Objective Functions

- What do we want from our weights?
  - Depends!
  - So far: minimize (training) errors:

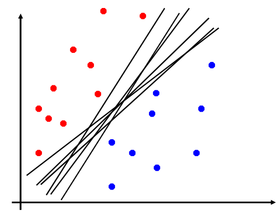
$$\sum_i step\left(\mathbf{w}^\top \mathbf{f}_i(\mathbf{y}_i^*) - \max_{\mathbf{y} \neq \mathbf{y}_i^*} \mathbf{w}^\top \mathbf{f}_i(\mathbf{y})\right)$$

$\mathbf{w}^\top \mathbf{f}_i(\mathbf{y}^i) - \max_{\mathbf{y} \neq \mathbf{y}_i^*} \mathbf{w}^\top \mathbf{f}_i(\mathbf{y})$

  - This is the “zero-one loss”
    - Discontinuous, minimizing is NP-complete
    - Not really what we want anyway
  - Maximum entropy and SVMs have other objectives related to zero-one loss

Linear Separators

- Which of these linear separators is optimal?



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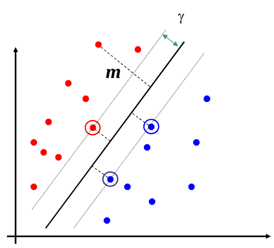
Classification Margin (Binary)

- Distance of $\mathbf{x}_i$ to separator is its margin, m_i
- Examples closest to the hyperplane are support vectors
- Margin γ of the separator is the minimum m



Classification Margin

- For each example $\mathbf{x}_i$ and possible mistaken candidate $\mathbf{y}$, we avoid that mistake by a margin $m_i(\mathbf{y})$ (with zero-one loss)

$$m_i(\mathbf{y}) = \mathbf{w}^\top \mathbf{f}_i(\mathbf{y}_i^*) - \mathbf{w}^\top \mathbf{f}_i(\mathbf{y})$$

- Margin γ of the entire separator is the minimum m

$$\gamma = \min_i \left(\mathbf{w}^\top \mathbf{f}_i(\mathbf{y}_i^*) - \max_{\mathbf{y} \neq \mathbf{y}_i^*} \mathbf{w}^\top \mathbf{f}_i(\mathbf{y})\right)$$

- It is also the largest γ for which the following constraints hold

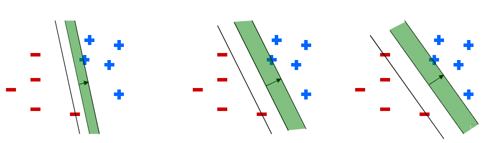
$$\forall i, \forall \mathbf{y} \quad \mathbf{w}^\top \mathbf{f}_i(\mathbf{y}_i^*) \geq \mathbf{w}^\top \mathbf{f}_i(\mathbf{y}) + \gamma \ell_i(\mathbf{y})$$

Maximum Margin

- Separable SVMs: find the max-margin w

$$\max_{||\mathbf{w}||=1} \gamma \qquad \ell_i(\mathbf{y}) = \begin{cases} 0 & \text{if } \mathbf{y} = \mathbf{y}_i^* \\ 1 & \text{if } \mathbf{y} \neq \mathbf{y}_i^* \end{cases}$$

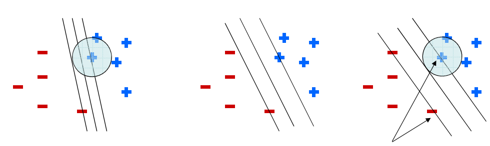
$$\forall i, \forall \mathbf{y} \quad \mathbf{w}^\top \mathbf{f}_i(\mathbf{y}_i^*) \geq \mathbf{w}^\top \mathbf{f}_i(\mathbf{y}) + \gamma \ell_i(\mathbf{y})$$



  - Can stick this into Matlab and (slowly) get an SVM
  - Won’t work (well) if non-separable

Why Max Margin?

- Why do this? Various arguments:
  - Solution depends only on the boundary cases, or *support vectors* (but remember how this diagram is broken!)
  - Solution robust to movement of support vectors
  - Sparse solutions (features not in support vectors get zero weight)
  - Generalization bound arguments
  - Works well in practice for many problems



Max Margin / Small Norm

- Reformulation: find the smallest w which separates data

Remember this condition?

$$\max_{||\mathbf{w}||=1} \gamma$$

$$\forall i, \mathbf{y} \quad \mathbf{w}^\top \mathbf{f}_i(\mathbf{y}_i^*) \ge \mathbf{w}^\top \mathbf{f}_i(\mathbf{y}) + \gamma \ell_i(\mathbf{y})$$

- γ scales linearly in w, so if ||w|| isn't constrained, we can take any separating w and scale up our margin

$$\gamma \;=\; \min_{i, \mathbf{y} \neq \mathbf{y}_i^*} [\mathbf{w}^\top \mathbf{f}_i(\mathbf{y}_i^*) - \mathbf{w}^\top \mathbf{f}_i(\mathbf{y})]/\ell_i(\mathbf{y})$$

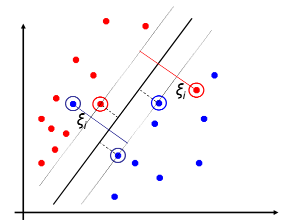
- Instead of fixing the scale of w, we can fix $\gamma = 1$

$$\min_{\mathbf{w}} \frac{1}{2}||\mathbf{w}||^2$$

$$\forall i, \mathbf{y} \quad \mathbf{w}^\top \mathbf{f}_i(\mathbf{y}_i^*) \ge \mathbf{w}^\top \mathbf{f}_i(\mathbf{y}) + 1\ell_i(\mathbf{y})$$

Soft Margin Classification

- What if the training set is not linearly separable?
- *Slack variables* ξ_i can be added to allow misclassification of difficult or noisy examples, resulting in a *soft margin* classifier



Maximum Margin

Note: exist other choices of how to penalize slacks!

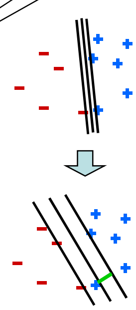
- Non-separable SVMs
  - Add slack to the constraints
  - Make objective pay (linearly) for slack:

$$\min_{\mathbf{w},\xi} \frac{1}{2}||\mathbf{w}||^2 + C\sum_i \xi_i$$

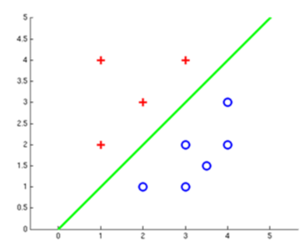
$$\forall i, \mathbf{y}, \quad \mathbf{w}^\top \mathbf{f}_i(\mathbf{y}_i^*) + \xi_i \ge \mathbf{w}^\top \mathbf{f}_i(\mathbf{y}) + \ell_i(\mathbf{y})$$

  - C is called the *capacity* of the SVM – the smoothing knob

- Learning:
  - Can still stick this into Matlab if you want
  - Constrained optimization is hard; better methods!
  - We'll come back to this later



Maximum Margin



Likelihood

Linear Models: Maximum Entropy

- Maximum entropy (logistic regression)
  - Use the scores as probabilities:

$$\mathsf{P}(\mathbf{y}|\mathbf{x}, \mathbf{w}) = \frac{\exp(\mathbf{w}^\top \mathbf{f}(\mathbf{y}))}{\sum_{\mathbf{y}'} \exp(\mathbf{w}^\top \mathbf{f}(\mathbf{y}'))}$$

Make positive

Normalize

  - Maximize the (log) conditional likelihood of training data

$$L(\mathbf{w}) = \log \prod_i \mathsf{P}(\mathbf{y}_i^*|\mathbf{x}_i, \mathbf{w}) = \sum_i \log \left(\frac{\exp(\mathbf{w}^\top \mathbf{f}_i(\mathbf{y}_i^*))}{\sum_{\mathbf{y}} \exp(\mathbf{w}^\top \mathbf{f}_i(\mathbf{y}))} \right)$$

$$= \sum_i \left(\mathbf{w}^\top \mathbf{f}_i(\mathbf{y}_i^*) - \log \sum_{\mathbf{y}} \exp(\mathbf{w}^\top \mathbf{f}_i(\mathbf{y})) \right)$$

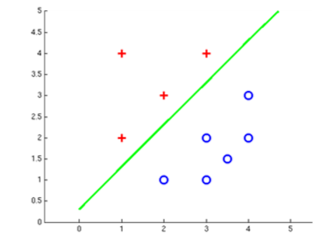
Maximum Entropy II

- Motivation for maximum entropy:
  - Connection to maximum entropy principle (sort of)
  - Might want to do a good job of being uncertain on noisy cases…
  - … in practice, though, posteriors are pretty peaked
- Regularization (smoothing)

$$\max_{\mathbf{w}} \sum_i \left(\mathbf{w}^\top \mathbf{f}_i(\mathbf{y}_i^*) - \log \sum_{\mathbf{y}} \exp(\mathbf{w}^\top \mathbf{f}_i(\mathbf{y})) \right) - k||\mathbf{w}||^2$$

$$\min_{\mathbf{w}} \; k||\mathbf{w}||^2 - \sum_i \left(\mathbf{w}^\top \mathbf{f}_i(\mathbf{y}_i^*) - \log \sum_{\mathbf{y}} \exp(\mathbf{w}^\top \mathbf{f}_i(\mathbf{y})) \right)$$

Maximum Entropy



Loss Comparison

Log-Loss

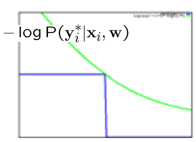
- If we view maxent as a minimization problem:

$$\min_{\mathbf{w}} \; k||\mathbf{w}||^2 + \sum_i - \left(\mathbf{w}^\top \mathbf{f}_i(\mathbf{y}_i^*) - \log \sum_{\mathbf{y}} \exp(\mathbf{w}^\top \mathbf{f}_i(\mathbf{y})) \right)$$

- This minimizes the “log loss” on each example

$$- \left(\mathbf{w}^\top \mathbf{f}_i(\mathbf{y}_i^*) - \log \sum_{\mathbf{y}} \exp(\mathbf{w}^\top \mathbf{f}_i(\mathbf{y})) \right) = - \log \mathsf{P}(\mathbf{y}_i^* | \mathbf{x}_i, \mathbf{w})$$

$$step \left(\mathbf{w}^\top \mathbf{f}_i(\mathbf{y}_i^*) - \max_{\mathbf{y} \neq \mathbf{y}_i^*} \mathbf{w}^\top \mathbf{f}_i(\mathbf{y}) \right)$$



- One view: log loss is an *upper bound* on zero-one loss

Remember SVMs…

- We had a constrained minimization

$$\min_{\mathbf{w},\xi} \frac{1}{2}||\mathbf{w}||^2 + C \sum_i \xi_i$$

$$\forall i, \mathbf{y}, \quad \mathbf{w}^\top \mathbf{f}_i(\mathbf{y}_i^*) + \xi_i \geq \mathbf{w}^\top \mathbf{f}_i(\mathbf{y}) + \ell_i(\mathbf{y})$$

- …but we can solve for ξ_i

$$\forall i, \mathbf{y}, \quad \xi_i \geq \mathbf{w}^\top \mathbf{f}_i(\mathbf{y}) + \ell_i(\mathbf{y}) - \mathbf{w}^\top \mathbf{f}_i(\mathbf{y}_i^*)$$

$$\forall i, \quad \xi_i = \max_{\mathbf{y}} \left(\mathbf{w}^\top \mathbf{f}_i(\mathbf{y}) + \ell_i(\mathbf{y}) \right) - \mathbf{w}^\top \mathbf{f}_i(\mathbf{y}_i^*)$$

- Giving

$$\min_{\mathbf{w}} \frac{1}{2}||\mathbf{w}||^2 + C \sum_i \left(\max_{\mathbf{y}} \left(\mathbf{w}^\top \mathbf{f}_i(\mathbf{y}) + \ell_i(\mathbf{y}) \right) - \mathbf{w}^\top \mathbf{f}_i(\mathbf{y}_i^*) \right)$$

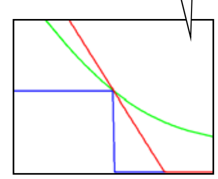
Hinge Loss

Plot really only right in binary case

- Consider the per-instance objective:

$$\min_{\mathbf{w}} \; k||\mathbf{w}||^2 + \sum_i \left(\max_{\mathbf{y}} \left(\mathbf{w}^\top \mathbf{f}_i(\mathbf{y}) + \ell_i(y) \right) - \mathbf{w}^\top \mathbf{f}_i(\mathbf{y}_i^*) \right)$$

- This is called the “hinge loss”
  - Unlike maxent / log loss, you stop gaining objective once the true label wins by enough
  - You can start from here and derive the SVM objective
  - Can solve directly with sub-gradient decent (e.g. Pegasos: Shalev-Shwartz et al 07)



$$\mathbf{w}^\top \mathbf{f}_i(\mathbf{y}_i^*) - \max_{\mathbf{y} \neq \mathbf{y}_i^*} \left(\mathbf{w}^\top \mathbf{f}_i(\mathbf{y}) \right)$$

Max vs "Soft-Max" Margin

- SVMs:

$$\min_{\mathbf{w}} \; k\|\mathbf{w}\|^2 - \sum_i \Big(\underbrace{\mathbf{w}^\top \mathbf{f}_i(\mathbf{y}_i^*) - \max_{\mathbf{y}} \left(\mathbf{w}^\top \mathbf{f}_i(\mathbf{y}) + \ell_i(\mathbf{y}) \right)}_{\text{You can make this zero}} \Big)$$

- Maxent:

$$\min_{\mathbf{w}} \; k\|\mathbf{w}\|^2 - \sum_i \Big(\underbrace{\mathbf{w}^\top \mathbf{f}_i(\mathbf{y}_i^*) - \log \sum_{\mathbf{y}} \exp\left(\mathbf{w}^\top \mathbf{f}_i(\mathbf{y}) \right)}_{\text{... but not this one}} \Big)$$

- Very similar! Both try to make the true score better than a function of the other scores
  - The SVM tries to beat the augmented runner-up
  - The Maxent classifier tries to beat the "soft-max"

Loss Functions: Comparison

- Zero-One Loss

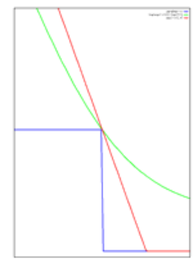
$$\sum_i step \left(\mathbf{w}^\top \mathbf{f}_i(\mathbf{y}_i^*) - \max_{\mathbf{y} \neq \mathbf{y}_i^*} \mathbf{w}^\top \mathbf{f}_i(\mathbf{y}) \right)$$

- Hinge

$$\sum_i \left(\mathbf{w}^\top \mathbf{f}_i(\mathbf{y}_i^*) - \max_{\mathbf{y}} \left(\mathbf{w}^\top \mathbf{f}_i(\mathbf{y}) + \ell_i(\mathbf{y}) \right) \right)$$

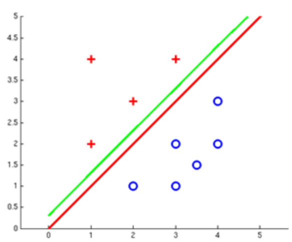
- Log

$$\sum_i \left(\mathbf{w}^\top \mathbf{f}_i(\mathbf{y}_i^*) - \log \sum_{\mathbf{y}} \exp\left(\mathbf{w}^\top \mathbf{f}_i(\mathbf{y}) \right) \right)$$



$\mathbf{w}^\top \mathbf{f}_i(\mathbf{y}_i^*) - \max_{\mathbf{y} \neq \mathbf{y}_i^*} \left(\mathbf{w}^\top \mathbf{f}_i(\mathbf{y}) \right)$

Separators: Comparison



Conditional vs Joint Likelihood

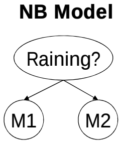
Example: Sensors

Reality

Raining | Sunny

P(+,+,r) = 3/8 | P(-,-,r) = 1/8 | P(+,+,s) = 1/8 | P(-,-,s) = 3/8

NB Model



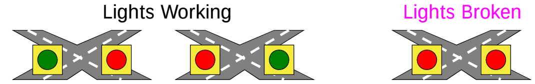
NB FACTORS:
- P(s) = 1/2
- P(+|s) = 1/4
- P(+|r) = 3/4

PREDICTIONS:
- P(r,+,+) = (½)(¾)(¾)
- P(s,+,+) = (½)(¼)(¼)
- P(r|+,+) = 9/10
- P(s|+,+) = 1/10

Example: Stoplights

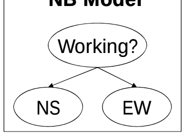
Reality

Lights Working | Lights Broken



P(g,r,w) = 3/7 | P(r,g,w) = 3/7 | P(r,r,b) = 1/7

NB Model



NB FACTORS:
- P(w) = 6/7
- P(r|w) = 1/2
- P(g|w) = 1/2

- P(b) = 1/7
- P(r|b) = 1
- P(g|b) = 0

Example: Stoplights

- What does the model say when both lights are red?
  - P(b,r,r) = (1/7)(1)(1) = 1/7 = 4/28
  - P(w,r,r) = (6/7)(1/2)(1/2) = 6/28 = 6/28
  - P(w|r,r) = 6/10!
- We'll guess that (r,r) indicates lights are working!
- Imagine if P(b) were boosted higher, to 1/2:
  - P(b,r,r) = (1/2)(1)(1) = 1/2 = 4/8
  - P(w,r,r) = (1/2)(1/2)(1/2) = 1/8 = 1/8
  - P(w|r,r) = 1/5!
- Changing the parameters bought accuracy at the expense of data likelihood

Duals and Kernels

Nearest-Neighbor Classification

- Nearest neighbor, e.g. for digits:
  - Take new example
  - Compare to all training examples
  - Assign based on closest example

- Encoding: image is vector of intensities:

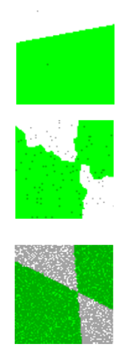
 $= \langle 0.0\ 0.0\ 0.3\ 0.8\ 0.7\ 0.1 \ldots 0.0 \rangle$

- Similarity function:
  - E.g. dot product of two images' vectors

$$\text{sim}(x, y) = x^\top y = \sum_i x_i y_i$$

Non-Parametric Classification

- Non-parametric: more examples means (potentially) more complex classifiers
- How about K-Nearest Neighbor?
  - We can be a little more sophisticated, averaging several neighbors
  - But, it's still not really error-driven learning
  - The magic is in the distance function
- Overall: we can exploit rich similarity functions, but not objective-driven learning



A Tale of Two Approaches…

- Nearest neighbor-like approaches
  - Work with data through similarity functions
  - No explicit "learning"
- Linear approaches
  - Explicit training to reduce empirical error
  - Represent data through features
- Kernelized linear models
  - Explicit training, but driven by similarity!
  - Flexible, powerful, very very slow

The Perceptron, Again

- Start with zero weights
- Visit training instances one by one
  - Try to classify

$$\hat{\mathbf{y}} = \arg\max_{\mathbf{y} \in \mathcal{Y}(\mathbf{x})} \mathbf{w}^\top \mathbf{f}_i(\mathbf{y})$$

  - If correct, no change!
  - If wrong: adjust weights

$$\mathbf{w} \leftarrow \mathbf{w} + \mathbf{f}_i(\mathbf{y}_i^*)$$
$$\mathbf{w} \leftarrow \mathbf{w} - \mathbf{f}_i(\hat{\mathbf{y}})$$
$$\mathbf{w} \leftarrow \mathbf{w} + (\mathbf{f}_i(\mathbf{y}_i^*) - \mathbf{f}_i(\hat{\mathbf{y}}))$$
$$\mathbf{w} \leftarrow \mathbf{w} + \boxed{\Delta_i(\hat{\mathbf{y}})}$$

mistake vectors

Perceptron Weights

- What is the final value of w? $\boxed{\mathbf{w} \leftarrow \mathbf{w} + \Delta_i(\mathbf{y})}$
  - Can it be an arbitrary real vector?
  - No! It's built by adding up feature vectors (mistake vectors).

$$\mathbf{w} = \Delta_i(\mathbf{y}) + \Delta_{i'}(\mathbf{y}') + \cdots$$

$$\mathbf{w} = \sum_{i,\mathbf{y}} \alpha_i(\mathbf{y})\Delta_i(\mathbf{y}) \quad \textit{mistake counts}$$

- Can reconstruct weight vectors (the primal representation) from update counts (the dual representation) for each i

$$\alpha_i = \langle \alpha_i(\mathbf{y}_1) \;\; \alpha_i(\mathbf{y}_2) \;\; \ldots \;\; \alpha_i(\mathbf{y}_n) \rangle$$

Dual Perceptron

$$\mathbf{w} = \sum_{i,\mathbf{y}} \alpha_i(\mathbf{y})\Delta_i(\mathbf{y})$$

- Track mistake counts rather than weights
- Start with zero counts (α)
- For each instance x
  - Try to classify

$$\hat{\mathbf{y}} = \arg\max_{\mathbf{y} \in \mathcal{Y}(\mathbf{x})} \mathbf{w}^\top \mathbf{f}(\mathbf{y})$$

$$\hat{\mathbf{y}} = \arg\max_{\mathbf{y} \in \mathcal{Y}(\mathbf{x}_i)} \sum_{i',\mathbf{y}'} \alpha_{i'}(\mathbf{y}')\Delta_{i'}(\mathbf{y}')^\top \mathbf{f}_i(\mathbf{y})$$

  - If correct, no change!
  - If wrong: raise the mistake count for this example and prediction

$$\alpha_i(\hat{\mathbf{y}}) \leftarrow \alpha_i(\hat{\mathbf{y}}) + 1 \qquad \mathbf{w} \leftarrow \mathbf{w} + \Delta_i(\hat{\mathbf{y}})$$

Dual / Kernelized Perceptron

- How to classify an example x?

$$\begin{aligned}
score(\mathbf{y}) = \mathbf{w}^\top \mathbf{f}_i(\mathbf{y}) &= \left(\sum_{i',\mathbf{y}'} \alpha_{i'}(\mathbf{y}')\Delta_{i'}(\mathbf{y}')\right)^\top \mathbf{f}_i(\mathbf{y}) \\
&= \sum_{i',\mathbf{y}'} \alpha_{i'}(\mathbf{y}')\left(\Delta_{i'}(\mathbf{y}')^\top \mathbf{f}_i(\mathbf{y})\right) \\
&= \sum_{i',\mathbf{y}'} \alpha_{i'}(\mathbf{y}')\left(\mathbf{f}_{i'}(\mathbf{y}^*_{i'})^\top \mathbf{f}_i(\mathbf{y}) - \mathbf{f}_{i'}(\mathbf{y}')^\top \mathbf{f}_i(\mathbf{y})\right) \\
&= \sum_{i',\mathbf{y}'} \alpha_{i'}(\mathbf{y}')\left(K(\mathbf{y}^*_{i'}, \mathbf{y}) - K(\mathbf{y}', \mathbf{y})\right)
\end{aligned}$$

- If someone tells us the value of K for each pair of candidates, never need to build the weight vectors

Issues with Dual Perceptron

- Problem: to score each candidate, we may have to compare to *all* training candidates

$$score(\mathbf{y}) = \sum_{i',\mathbf{y}'} \alpha_{i'}(\mathbf{y}')\left(K(\mathbf{y}^*_{i'}, \mathbf{y}) - K(\mathbf{y}', \mathbf{y})\right)$$

  - Very, very slow compared to primal dot product!
  - One bright spot: for perceptron, only need to consider candidates we made mistakes on during training
  - Slightly better for SVMs where the alphas are (in theory) sparse
- This problem is serious: fully dual methods (including kernel methods) tend to be extraordinarily slow
- Of course, we can (so far) also accumulate our weights as we go...

Kernels: Who Cares?

- So far: a very strange way of doing a very simple calculation
- "Kernel trick": we can substitute any* similarity function in place of the dot product
- Lets us learn new kinds of hypotheses

* Fine print: if your kernel doesn't satisfy certain technical requirements, lots of proofs break. E.g. convergence, mistake bounds. In practice, illegal kernels *sometimes* work (but not always).

Some Kernels

- Kernels implicitly map original vectors to higher dimensional spaces, take the dot product there, and hand the result back
- Linear kernel: $K(x, x') = x' \cdot x' = \sum_i x_i x_i'$
- Quadratic kernel: $K(x, x') = (x \cdot x' + 1)^2$

$$= \sum_{i,j} x_i x_j x_i' x_j' + 2\sum_i x_i x_i' + 1$$

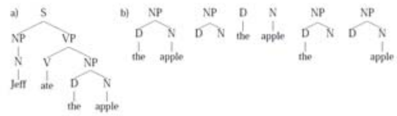
- RBF: infinite dimensional representation

$$K(x, x') = \exp(-||x - x'||^2)$$

- Discrete kernels: e.g. string kernels, tree kernels

Tree Kernels

[Collins and Duffy 01]



- Want to compute number of common subtrees between T, T'
- Add up counts of all pairs of nodes n, n'
  - Base: if n, n' have different root productions, or are depth 0:

$$C(n_1, n_2) = 0$$

  - Base: if n, n' are share the same root production:

$$C(n_1, n_2) = \lambda \prod_{j=1}^{nc(n_1)} (1 + C(ch(n_1, j), ch(n_2, j)))$$

44

Dual Formulation for SVMs

- We want to optimize: (separable case for now)

$$\min_{\mathbf{w}} \quad \frac{1}{2}||\mathbf{w}||^2$$
$$\forall i, \mathbf{y} \quad \mathbf{w}^\top \mathbf{f}_i(\mathbf{y}_i^*) \geq \mathbf{w}^\top \mathbf{f}_i(\mathbf{y}) + \ell_i(\mathbf{y})$$

- This is hard because of the constraints
- Solution: method of Lagrange multipliers
- The *Lagrangian* representation of this problem is:

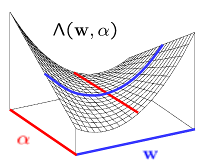
$$\min_{\mathbf{w}} \max_{\alpha \geq 0} \quad \Lambda(\mathbf{w}, \alpha) = \frac{1}{2}||\mathbf{w}||^2 - \sum_{i,\mathbf{y}} \alpha_i(\mathbf{y}) \left(\mathbf{w}^\top \mathbf{f}_i(\mathbf{y}_i^*) - \mathbf{w}^\top \mathbf{f}_i(\mathbf{y}) - \ell_i(\mathbf{y})\right)$$

- All we've done is express the constraints as an adversary which leaves our objective alone if we obey the constraints but ruins our objective if we violate any of them

Lagrange Duality

- We start out with a constrained optimization problem:

$$f(\mathbf{w}^*) = \min_{\mathbf{w}} f(\mathbf{w})$$
$$g(\mathbf{w}) \geq 0$$



- We form the Lagrangian:

$$\Lambda(\mathbf{w}, \alpha) = f(\mathbf{w}) - \alpha\, g(\mathbf{w})$$

- This is useful because the constrained solution is a saddle point of Λ (this is a general property):

$$f(\mathbf{w}^*) = \underbrace{\min_{\mathbf{w}} \max_{\alpha \geq 0} \Lambda(\mathbf{w}, \alpha)}_{\text{Primal problem in } \mathbf{w}} = \underbrace{\max_{\alpha \geq 0} \min_{\mathbf{w}} \Lambda(\mathbf{w}, \alpha)}_{\text{Dual problem in } \alpha}$$

Dual Formulation II

- Duality tells us that

$$\min_{\mathbf{w}} \max_{\alpha \geq 0} \quad \frac{1}{2}||\mathbf{w}||^2 - \sum_{i,\mathbf{y}} \alpha_i(\mathbf{y}) \left(\mathbf{w}^\top \mathbf{f}_i(\mathbf{y}_i^*) - \mathbf{w}^\top \mathbf{f}_i(\mathbf{y}) - \ell_i(\mathbf{y})\right)$$

has the same value as

$$\max_{\alpha \geq 0} \overbrace{\min_{\mathbf{w}} \quad \frac{1}{2}||\mathbf{w}||^2 - \sum_{i,\mathbf{y}} \alpha_i(\mathbf{y}) \left(\mathbf{w}^\top \mathbf{f}_i(\mathbf{y}_i^*) - \mathbf{w}^\top \mathbf{f}_i(\mathbf{y}) - \ell_i(\mathbf{y})\right)}^{Z(\alpha)}$$

- This is useful because if we think of the α's as constants, we have an unconstrained min in w that we can solve analytically.
- Then we end up with an optimization over α instead of $\mathbf{w}$ (easier).

Dual Formulation III

- Minimize the Lagrangian for fixed α's:

$$\Lambda(\mathbf{w}, \alpha) = \frac{1}{2}||\mathbf{w}||^2 - \sum_{i,\mathbf{y}} \alpha_i(\mathbf{y}) \left(\mathbf{w}^\top \mathbf{f}_i(\mathbf{y}_i^*) - \mathbf{w}^\top \mathbf{f}_i(\mathbf{y}) - \ell_i(\mathbf{y})\right)$$

$$\frac{\partial \Lambda(\mathbf{w}, \alpha)}{\partial \mathbf{w}} = \mathbf{w} - \sum_{i,\mathbf{y}} \alpha_i(\mathbf{y}) \left(\mathbf{f}_i(\mathbf{y}_i^*) - \mathbf{f}_i(\mathbf{y})\right)$$

$$\frac{\partial \Lambda(\mathbf{w}, \alpha)}{\partial \mathbf{w}} = 0 \quad \Rightarrow \quad \mathbf{w} = \sum_{i,\mathbf{y}} \alpha_i(\mathbf{y}) \left(\mathbf{f}_i(\mathbf{y}_i^*) - \mathbf{f}_i(\mathbf{y})\right)$$

- So we have the Lagrangian as a function of only α's:

$$\min_{\alpha \geq 0} Z(\alpha) = \frac{1}{2}\left|\left|\sum_{i,\mathbf{y}} \alpha_i(\mathbf{y}) \left(\mathbf{f}_i(\mathbf{y}_i^*) - \mathbf{f}_i(\mathbf{y})\right)\right|\right|^2 - \sum_{i,\mathbf{y}} \alpha_i(\mathbf{y}) \ell_i(\mathbf{y})$$

Back to Learning SVMs

- We want to find α which minimize

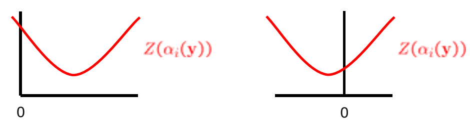
$$\min_{\alpha \geq 0} \Lambda(\alpha) = \frac{1}{2}\left|\left|\sum_{i,\mathbf{y}} \alpha_i(\mathbf{y}) \left(\mathbf{f}_i(\mathbf{y}^i) - \mathbf{f}_i(\mathbf{y})\right)\right|\right|^2 - \sum_{i,\mathbf{y}} \alpha_i(\mathbf{y}) \ell_i(\mathbf{y})$$
$$\forall i, \quad \sum_{\mathbf{y}} \alpha_i(\mathbf{y}) = C$$

- This is a quadratic program:
  - Can be solved with general QP or convex optimizers
  - But they don't scale well to large problems
  - Cf. maxent models work fine with general optimizers (e.g. CG, L-BFGS)
- How would a special purpose optimizer work?

Coordinate Descent I

$$\min_{\alpha \geq 0} Z(\alpha) = \min_{\alpha \geq 0} \frac{1}{2} \left\| \sum_{i,\mathbf{y}} \alpha_i(\mathbf{y}) \left(\mathbf{f}_i(\mathbf{y}_i^*) - \mathbf{f}_i(\mathbf{y})\right) \right\|^2 - \sum_{i,\mathbf{y}} \alpha_i(\mathbf{y}) \ell_i(\mathbf{y})$$

- Despite all the mess, Z is just a quadratic in each $\alpha_i(\mathbf{y})$
- Coordinate descent: optimize one variable at a time



- If the unconstrained argmin on a coordinate is negative, just clip to zero…

Coordinate Descent II

- Ordinarily, treating coordinates independently is a bad idea, but here the update is very fast and simple

$$\alpha_i(\mathbf{y}) \leftarrow \max\left(0, \alpha_i(\mathbf{y}) + \frac{\ell_i(\mathbf{y}) - \mathbf{w}^\top \left(\mathbf{f}_i(\mathbf{y}_i^*) - \mathbf{f}_i(\mathbf{y})\right)}{\left\|\left(\mathbf{f}_i(\mathbf{y}_i^*) - \mathbf{f}_i(\mathbf{y})\right)\right\|^2}\right)$$

- So we visit each axis many times, but each visit is quick

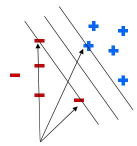
- This approach works fine for the separable case
- For the non-separable case, we just gain a simplex constraint and so we need slightly more complex methods (SMO, exponentiated gradient)

$$\forall i, \quad \sum_{\mathbf{y}} \alpha_i(\mathbf{y}) = C$$

What are the Alphas?

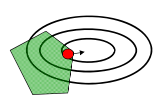
- Each candidate corresponds to a primal constraint

$$\min_{\mathbf{w},\xi} \quad \frac{1}{2}\|\mathbf{w}\|^2 + C\sum_i \xi_i$$
$$\forall i, \mathbf{y} \quad \mathbf{w}^\top \mathbf{f}_i(\mathbf{y}_i^*) \geq \mathbf{w}^\top \mathbf{f}_i(\mathbf{y}) + \ell_i(\mathbf{y}) - \xi_i$$



Support vectors

- In the solution, an $\alpha_i(\mathbf{y})$ will be:
  - Zero if that constraint is inactive
  - Positive if that constrain is active
  - i.e. positive on the support vectors



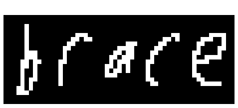
- Support vectors contribute to weights:

$$\mathbf{w} = \sum_{i,\mathbf{y}} \alpha_i(\mathbf{y}) \left(\mathbf{f}_i(\mathbf{y}_i^*) - \mathbf{f}_i(\mathbf{y})\right)$$

Structure

Handwriting recognition

x **y**



brace

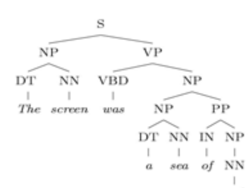
Sequential structure

[Slides: Taskar and Klein 05]

CFG Parsing

x **y**

The screen was a sea of red



Recursive structure

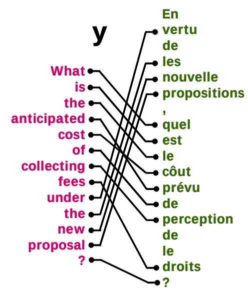
Bilingual word alignment

x

What is the anticipated cost of collecting fees under the new proposal?

En vertu de nouvelle propositions, quel est le côût prévu de perception de les droits?

y



Combinatorial structure

Structured Models

$$prediction(\mathbf{x}, \mathbf{w}) = \arg\max_{\mathbf{y} \in \mathcal{Y}(\mathbf{x})} score(\mathbf{y}, \mathbf{w})$$

space of feasible outputs

Assumption:

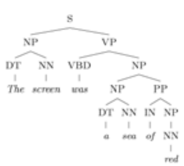
$$score(\mathbf{y}, \mathbf{w}) = \mathbf{w}^\top \mathbf{f}(\mathbf{y}) = \sum_p \mathbf{w}^\top \mathbf{f}(\mathbf{y}_p)$$

Score is a sum of local "part" scores

Parts = nodes, edges, productions

CFG Parsing

$$P(\mathbf{y} \mid \mathbf{x}) \propto \prod_{A \to \alpha \in (\mathbf{x},\mathbf{y})} \phi(A \to \alpha)$$



$\mathbf{f} : \mathcal{X} \times \mathcal{Y} \to \Re^d$

#(NP → DT NN)

...

#(PP → IN NP)

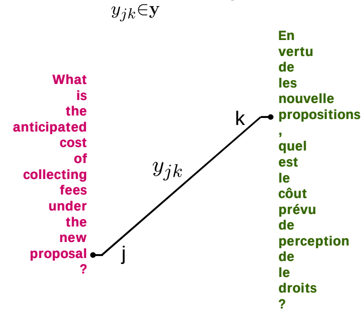
...

#(NN → 'sea')

$$\prod_{A \to \alpha \in (\mathbf{x},\mathbf{y})} \exp\left\{\mathbf{w}^\top \mathbf{f}(A \to \alpha)\right\} = \exp\{\mathbf{w}^\top \mathbf{f}(\mathbf{x}, \mathbf{y})\}$$

Bilingual word alignment

$$\sum_{y_{jk} \in \mathbf{y}} \mathbf{w}^\top \mathbf{f}(\mathbf{x}_{jk}) = \mathbf{w}^\top \mathbf{f}(\mathbf{x}, \mathbf{y})$$

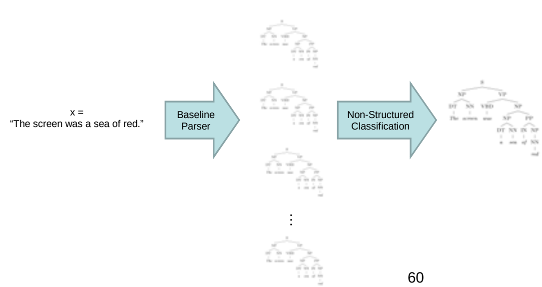


$\mathbf{f}(\mathbf{x}_{jk})$

- association
- position
- orthography

Option 0: Reranking

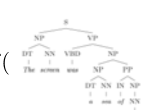
[e.g. Charniak and Johnson 05]



60

Reranking

- Advantages:
  - Directly reduce to non-structured case
  - No locality restriction on features

$\mathbf{f}($  $) =$

- Disadvantages:
  - Stuck with errors of baseline parser
  - Baseline system must produce n-best lists
  - But, feedback is possible [McCloskey, Charniak, Johnson 2006]

Efficient Primal Decoding

- Common case: you have a black box which computes

$$\text{prediction}(\mathbf{x}) = \arg\max_{\mathbf{y}\in\mathcal{Y}(\mathbf{x})} \mathbf{w}^\top \mathbf{f}(\mathbf{y})$$

at least approximately, and you want to learn w

- Many learning methods require more (expectations, dual representations, k-best lists), but the most commonly used options do not
- Easiest option is the structured perceptron [Collins 01]
  - Structure enters here in that the search for the best y is typically a combinatorial algorithm (dynamic programming, matchings, ILPs, A*…)
  - Prediction is structured, learning update is not

62

Structured Margin

- Remember the margin objective:

$$\min_{\mathbf{w}} \quad \frac{1}{2}||\mathbf{w}||^2$$
$$\forall i, \mathbf{y} \quad \mathbf{w}^\top \mathbf{f}_i(\mathbf{y}_i^*) \geq \mathbf{w}^\top \mathbf{f}_i(\mathbf{y}) + \ell_i(\mathbf{y})$$

- This is still defined, but lots of constraints

Full Margin: OCR

- We want:

$$\arg\max_{\mathbf{y}} \mathbf{w}^\top \mathbf{f}(\text{brace}, \mathbf{y}) = \text{“brace”}$$

- Equivalently:

$$\mathbf{w}^\top \mathbf{f}(\text{brace}, \text{“brace”}) > \mathbf{w}^\top \mathbf{f}(\text{brace}, \text{“aaaaa”})$$
$$\mathbf{w}^\top \mathbf{f}(\text{brace}, \text{“brace”}) > \mathbf{w}^\top \mathbf{f}(\text{brace}, \text{“aaaab”})$$
$$\ldots$$
$$\mathbf{w}^\top \mathbf{f}(\text{brace}, \text{“brace”}) > \mathbf{w}^\top \mathbf{f}(\text{brace}, \text{“zzzzz”})$$

a lot!

Parsing example

- We want:

$$\arg\max_{\mathbf{y}} \mathbf{w}^\top \mathbf{f}(\text{‘It was red’}, \mathbf{y}) =$$

- Equivalently:

$$\mathbf{w}^\top \mathbf{f}(\text{‘It was red’}, \ ) > \mathbf{w}^\top \mathbf{f}(\text{‘It was red’}, \ )$$
$$\mathbf{w}^\top \mathbf{f}(\text{‘It was red’}, \ ) > \mathbf{w}^\top \mathbf{f}(\text{‘It was red’}, \ )$$
$$\ldots$$
$$\mathbf{w}^\top \mathbf{f}(\text{‘It was red’}, \ ) > \mathbf{w}^\top \mathbf{f}(\text{‘It was red’}, \ )$$

a lot!

Alignment example

- We want:

$$\arg\max_{\mathbf{y}} \mathbf{w}^\top \mathbf{f}(\text{‘What is the’ ‘Quel est le’}, \mathbf{y}) = $$

- Equivalently:

$$\mathbf{w}^\top \mathbf{f}(\text{‘What is the’ ‘Quel est le’}, \ ) > \mathbf{w}^\top \mathbf{f}(\text{‘What is the’ ‘Quel est le’}, \ )$$
$$\mathbf{w}^\top \mathbf{f}(\text{‘What is the’ ‘Quel est le’}, \ ) > \mathbf{w}^\top \mathbf{f}(\text{‘What is the’ ‘Quel est le’}, \ )$$
$$\ldots$$
$$\mathbf{w}^\top \mathbf{f}(\text{‘What is the’ ‘Quel est le’}, \ ) > \mathbf{w}^\top \mathbf{f}(\text{‘What is the’ ‘Quel est le’}, \ )$$

a lot!

Cutting Plane

- A constraint induction method [Joachims et al 09]
  - Exploits that the number of constraints you actually need per instance is typically very small
  - Requires (loss-augmented) primal-decode only
- Repeat:
  - Find the most violated constraint for an instance:

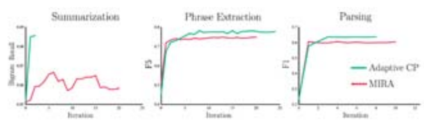
$$\forall \mathbf{y} \quad \mathbf{w}^\top \mathbf{f}_i(\mathbf{y}_i^*) \geq \mathbf{w}^\top \mathbf{f}_i(\mathbf{y}) + \ell_i(\mathbf{y})$$

$$\arg\max_{\mathbf{y}} \mathbf{w}^\top \mathbf{f}_i(\mathbf{y}) + \ell_i(\mathbf{y})$$

  - Add this constraint and resolve the (non-structured) QP (e.g. with SMO or other QP solver)

Cutting Plane

- Some issues:
  - Can easily spend too much time solving QPs
  - Doesn't exploit shared constraint structure
  - In practice, works pretty well; fast like MIRA, more stable, no averaging



M3Ns

- Another option: express all constraints in a packed form
  - Maximum margin Markov networks [Taskar et al 03]
  - Integrates solution structure deeply into the problem structure
- Steps
  - Express inference over constraints as an LP
  - Use duality to transform minimax formulation into min-min
  - Constraints factor in the dual along the same structure as the primal; alphas essentially act as a dual "distribution"
  - Various optimization possibilities in the dual

Likelihood, Structured

$$L(\mathbf{w}) = -k\|\mathbf{w}\|^2 + \sum_i \left(\mathbf{w}^\top \mathbf{f}_i(\mathbf{y}_i^*) - \log \sum_{\mathbf{y}} \exp(\mathbf{w}^\top \mathbf{f}_i(\mathbf{y})) \right)$$

$$\frac{\partial L(\mathbf{w})}{\partial \mathbf{w}} = -2k\mathbf{w} + \sum_i \left(\mathbf{f}_i(\mathbf{y}_i^*) - \sum_{\mathbf{y}} P(\mathbf{y}|\mathbf{x}_i)\mathbf{f}_i(\mathbf{y}) \right)$$

- Structure needed to compute:
  - Log-normalizer
  - Expected feature counts
    - E.g. if a feature is an indicator of DT-NN then we need to compute posterior marginals P(DT-NN|sentence) for each position and sum
- Also works with latent variables (more later)