Solving Large-Scale Granular Resource Allocation Problems Efficiently with POP

Deepak Narayanan[⋆], Fiodar Kazhamiaka[⋆], Firas Abuzaid[⋆], Peter Kraft[⋆], Akshay Agrawal[⋆], Srikanth Kandula[†], Stephen Boyd[⋆], Matei Zaharia[⋆]
[⋆]*Stanford University* [†]*Microsoft Research*

Abstract

Resource allocation problems in many computer systems can be formulated as mathematical optimization problems. However, finding exact solutions to these problems using off-the-shelf solvers is often intractable for large problem sizes with tight SLAs, leading system designers to rely on cheap, heuristic algorithms. We observe, however, that many allocation problems are *granular*: they consist of a large number of clients and resources, each client requests a small fraction of the total number of resources, and clients can interchangeably use different resources. For these problems, we propose an alternative approach that *reuses* the original optimization problem formulation and leads to better allocations than domain-specific heuristics. Our technique, Partitioned Optimization Problems (POP), *randomly* splits the problem into smaller problems (with a subset of the clients and resources in the system) and coalesces the resulting sub-allocations into a global allocation for all clients. We provide theoretical and empirical evidence as to why random partitioning works well. In our experiments, POP achieves allocations within 1.5% of the optimal with orders-of-magnitude improvements in runtime compared to existing systems for cluster scheduling, traffic engineering, and load balancing.

CCS Concepts. • **Networks** → **Traffic engineering algorithms**; *Network resources allocation*; • **Computer systems organization** → **Cloud computing**; • **Theory of computation** → *Scheduling algorithms.*

Keywords. Resource scheduling, optimization problems in computer systems, cluster scheduling, traffic engineering, load balancing.

SOSP '21, October 26–29, 2021, Virtual Event, Germany
© 2021 Copyright held by the owner/author(s). Publication rights licensed to ACM.
ACM ISBN 978-1-4503-8709-5/21/10…$15.00
https://doi.org/10.1145/3477132.3483588

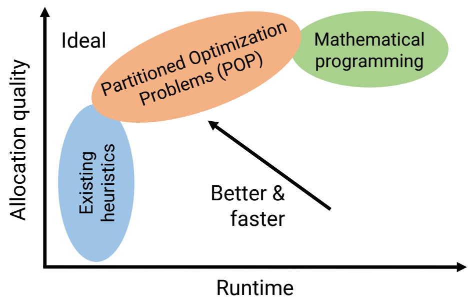


Figure 1. Tradeoff space between allocation quality (objective-dependent) and runtime. Our proposed technique (POP) is faster than directly solving mathematical programs, and computes better allocations than existing heuristic algorithms.

1 Introduction

As workloads become more computationally expensive and computer systems become larger, it has become common for systems to be shared among multiple users. As a result, deciding how resources (e.g., GPUs, links, servers) should be shared amongst various clients while optimizing for many macro-objectives is important across a number of domains (e.g., cluster scheduling, traffic engineering, load balancing).

Resource allocation problems can often be formulated as mathematical optimization programs [5, 19, 27, 29, 32, 33, 38, 40, 45]; the output of these programs is the allocation of resources (e.g., accelerators, servers, or network links) to each client (e.g., jobs, data shards, or traffic commodities). Unfortunately, solving these mathematical programs can be computationally expensive (Figure 1). The worst-case complexity for linear programs is approximately $O(n^{2.373})$ [11, 28], where n is the number of problem variables (even though LPs can often be solved faster depending on problem structure), and integer-linear programs are even more expensive. Mathematical programs for resource allocation can have millions of variables (e.g., one variable for every <client, resource> pair) for large-scale systems, leading to long solution times depending on the numerical solver used (e.g., 8 minutes for a cluster with 1000 jobs using SCS [34, 35]). Moreover, allocations often need to be recomputed frequently to keep up with dynamic changes in the system. Consequently, production systems such as B4 and BwE [23, 27] for traffic engineering, the Accordion load balancer [42] for distributed databases, and the Gavel job scheduler [32], hit performance bottlenecks when the numbers of clients and resources increase.

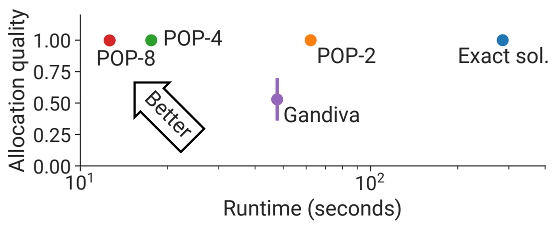


Figure 2. Comparison of Gavel's fair-sharing policy compared to its POP variants and Gandiva [49] on a GPU cluster. The scatterplot shows runtimes and mean allocation quality across 2048 jobs on a cluster with 1536 GPUs. POP-k uses k sub-problems.

Thus, the conventional wisdom in the systems community is that solving these programs *directly* often takes too long. Instead, production systems and researchers frequently use heuristics that are cheaper to compute. It is common to see some version of the following statement in a paper:

> "Since these algorithms take a long time, they are not practical for real-world deployments. Instead, they provide a baseline with which to compare faster approximation algorithms." – Taft [45].

The partition-placement algorithm in E-Store [45], the space-sharing-aware policy in Gandiva [49], and cluster management policies to allocate resources to containers in systems like Kubernetes [2], DRS [22], and OpenShift [3] all rely on heuristics. However, prior work shows that these heuristics are hard to maintain as problems scale and inputs change [44], are far from optimal (Figures 2, 9, and 13), and often do not extend to slightly modified objectives.

Although it seems that large optimization problems are too expensive to solve directly, we observe that many allocation problems in computer systems share several exploitable properties: the number of clients and resources is large, each client requests a small fraction of the total number of resources, and resources are fungible or substitutable (i.e., a job can make similar progress using *different* resources). For any such *granular* allocation problem, we propose POP, which stands for Partitioned Optimization Problems: a method for quickly computing allocations by *reusing* the original optimization problem formulation on subsets of the input. On several granular optimization problems, we found that POP can give close-to-optimal results with orders-of-magnitude faster runtimes than the full formulations. Importantly, since POP reuses the original problem formulations, it can be implemented in only a few lines of code.

The simplest way to apply POP is to divide clients and resources among k identical copies of the given optimization problem (each with a subset of the clients and resources). Each sub-problem has fewer equations and variables, leading to a super-linear runtime speedup. The sub-problems can also be executed in parallel. The overall allocation is a union of the allocations from the individual sub-problems. Our results show that randomly and evenly dividing clients and resources among sub-problems works well when clients are numerous and individually use only a small fraction of all resources. Empirically, we show that POP's resource allocations are nearly optimal on several optimization problems, including using real-world inputs. We also prove that the probability of a large optimality gap is small given an allocation problem with certain simple properties. POP has structural similarity with the first step of "primal decomposition" in convex optimization [10], but can be applied to a broader set of problems than those amenable to primal decomposition (separable objective, coupled constraints). We note that there can be other ways to POP an optimization problem, but these are beyond the scope of this paper.

In the wild, allocation problems do not always fit the definition of granular as presented above, e.g., a traffic engineering problem could have "large" clients (commodities) with substantial bandwidth demand, or a client might have to use a particular resource (e.g., link between two sites). Fortunately, in some such cases, we can transform the problem into a granular problem using two *granularization* techniques: *client splitting* and *resource splitting*. The "large" clients, which individually require a sizable fraction of total resources, can be split into multiple virtual clients who each receive partial allocations from multiple sub-problems. Since the number of "large" clients is small, by definition, POP's sub-problems remain small and still achieve a sizable runtime speedup. Similarly, resources can be split into multiple virtual resources, each with a fraction of the full resource's capacity.

POP cannot be applied to every allocation problem in systems because some problems are not granular or require a non-trivial partitioning into sub-problems (e.g., due to constraints). We discuss examples of such problems in §4.4.

Nevertheless, we found that POP is effective on a wide range of important problems in recent computer systems research. We evaluated POP on 6 different allocation objectives across three domains (cluster scheduling, traffic engineering, and load balancing). POP achieves empirical runtime improvements of up to 100× compared to the original optimization problem formulations while staying within 1.5% of optimal, and even up to 20× faster and 1.9× higher quality than heuristics. We integrated POP into real systems like Gavel, and found that downstream metrics like average job completion time and makespan are unaffected by using POP. We also found granularization useful in using POP to compute high-quality allocations for initially non-granular problems, like traffic engineering problems with a few large flows and links between specific sites. Our implementation is available at https://github.com/stanford-futuredata/POP.

2 Granular Allocation Problems

Computer systems are often shared among *clients* from multiple users (e.g., jobs in a cluster scheduler, commodities in a Wide Area Network). These clients might then request *resources* (e.g., GPUs or link capacity) from a central resource allocator, which determines how to map resources to clients. Resource allocation problems have three main components:

- **Search Space of Allocations:** Allocations specify how resources should be shared between clients. In cluster scheduling, an allocation can specify the fraction of wall-clock time each active job should spend on different types of resources (e.g., types of GPUs like K80, P100, V100, A100). In traffic engineering, an allocation can specify the flow each commodity should receive on different links. Allocations can also reason through the interactions between clients on different resources (e.g., the time fractions *pairs* of jobs should spend on various resources [32, 49]).
- **Objectives:** The objective that an optimization problem maximizes or minimizes is a function over the allocation, and specifies the metric (e.g., dollar cost, total flow) that needs to be optimized in solving the allocation problem. We observe that these functions are typically a max or sum over functions of per-client allocations, but can be other arbitrary functions as well. Convex functions are generally easier to optimize.
- **Constraints:** Most allocation problems also specify constraints to ensure that both clients and resources are not over-allocated (e.g., the total time fraction given to a single job across resource types cannot exceed 1.0) and that various invariants are maintained. These are specified as functions over the allocation A.

The goal of a resource allocation problem is to find the allocation value that is feasible (respects the provided constraints) and optimizes the provided objective.

We can then say an allocation problem is **granular** if:

- **Condition 1:** The number of clients and resources is large (on the order of 100s or more).
- **Condition 2:** Each client requests an insignificant fraction (e.g., $< 1\%$) of the total available resources.
- **Condition 3:** Resources are fungible or substitutable. In other words, if a client c is given resource r as part of an allocation A, there are multiple other resources $r' \neq r$ such that switching c to r' gives an allocation A' with similar objective value ($f(A) \approx f(A')$).
- **Condition 4:** If the resource allocation problem considers interactions between multiple clients (e.g., two jobs on the same server), then client combinations should be fungible or substitutable too.

As we show in §4, resource allocation problems in a number of different domains like cluster scheduling, traffic engineering, and load balancing, are granular. Furthermore, in certain cases, problems that violate some of these conditions can be made granular through granularization transformations (client and resource splitting in §3.3).

For example, in Gavel [32], a cluster scheduler for machine learning training workloads on clusters of GPUs, each job (client) requests a prescribed number of a resource (e.g., a specific kind of GPU) to make progress. Each job requests a small fraction of the total number of GPUs available in the cluster, and can be run on different types of GPUs with varying efficiencies. Additionally, when used with space sharing [32, 49], each job can be run with many other jobs (again with varying efficiencies). We assume that dependencies that specify when jobs are runnable are handled by a separate DAG scheduler. This is standard in systems such as Spark and Hadoop [50]. Such cross-job "when can job X run" dependencies are not under the purview of the resource schedulers considered in this paper, which try to determine how resources should be shared among already *runnable* jobs only.

In traffic engineering setups such as those considered in NCFlow [5], the clients are commodities, each resource is a network link between two sites in the Wide Area Network, and each commodity typically requests a small fraction of the total available capacity.

In load balancing, the clients are data shards, the resources are servers, and each shard can be handled by a small fraction of the total number of servers available in the system.

3 Partitioned Optimization Problems

Granular resource allocation problems can be split into sub-problems, where each sub-problem has a subset of the clients and resources in the full allocation problem. We leverage the large number of clients and resources to randomly partition clients and resources into sub-problems; this procedure yields high-quality allocations due to the law of large numbers. We call this technique Partitioned Optimization Problems (or POP for short). In the rest of this section, we describe the intuition, procedure, and benefits of POP.

3.1 Intuition

Optimization problems for large systems take a long time to solve in part because they have many variables. For example, consider an optimization problem that involves scheduling n jobs on m cloud VMs. Each VM has varying amounts of resources (e.g., CPU cores, GPUs, and RAM). To express the possibility of any job being assigned to any VM, an $n \times m$ matrix of variables would be needed; for 10^4 jobs and 10^4 VMs, the problem has 10^8 variables. Contemporary solvers often take hours to solve such problems, although the exact runtime depends on problem properties such as sparsity [48].

We can achieve much faster allocation computation times by decomposing the problem; for example, the problem of scheduling 10^3 jobs on 10^3 VMs ($100\times$ fewer variables) is much more tractable. This procedure of breaking up the larger problem into sub-problems *reduces the search space* explored by the solver, since interactions between all combinations of clients and resources are no longer considered.

Algorithm 1 POP Procedure.

```
**Input:** Clients and their attributes $X = [x_1, x_2, \dots, x_n]$, resources and their attributes $Y = [y_1, y_2, \dots, y_m]$, function to compute allocations GET_ALLOCATION : $(X, Y) \rightarrow A$, number of partitions $k$, (optional) splitting attribute $s$, (optional) ratio of extra virtual clients allowed $t$.
**Return:** Allocation for all $n$ clients, $A$.

// Optional: make the problem granular if it is not already.
$X' =$ SPLIT_CLIENTS$(X, s, t)$, $Y' =$ SPLIT_RESOURCES$(Y)$

// This is the partition step.
$[X'_1, X'_2, \dots, X'_k], [Y'_1, Y'_2, \dots, Y'_k] =$ partition$(X', Y', k)$

// This is the map step, can be performed in parallel.
for $i$ in range$(k)$ do
    $A_i =$ GET_ALLOCATION$(X'_i, Y'_i)$
end for

// This is the reduce step; allocations $A_i$ are combined.
$A =$ COALESCE$([A_1, A_2, \dots, A_k])$
```

Instead, only combinations of *subsets* of clients and resources are considered, which reduces runtime but also can reduce the quality of the allocation. In light of this, the interaction between clients and resources needs to be considered carefully to take into account the many global constraints in the original problem, as well as the objective (e.g., fairness). We find that on large granular resource allocation problems, splitting clients *randomly* and assigning an equal number of resources among sub-problems reduces the search space of feasible solutions that needs to be considered by solvers, while still ensuring that *some* high-quality feasible points are in the explored search space. This is the main intuition that allows POP to be effective, returning allocations of similar quality as the original formulation but faster.

3.2 Procedure for Granular Problems

The first step of POP is to **partition** larger allocation problems into smaller allocation sub-problems. The type of partitioning allowed is dependent on the objective and constraints of the allocation problem, and has implications on the runtime speedups and quality of the returned allocation. We can then re-use the map-reduce API [14, 50] (or divide-and-conquer): each of these sub-problems can be solved in parallel (**map** step), and then allocations from the sub-problems can be reconciled into a larger allocation for the entire problem (**reduce** step). We show pseudocode for this in Algorithm 1.

The partitioning step affects the runtime, the reconciliation complexity, and ultimately the quality of the final allocation. One straightforward approach that we explore in this paper is to divide *both* clients (e.g., jobs, shards, flows) and resources (e.g., servers, links) randomly into *sub-systems*,

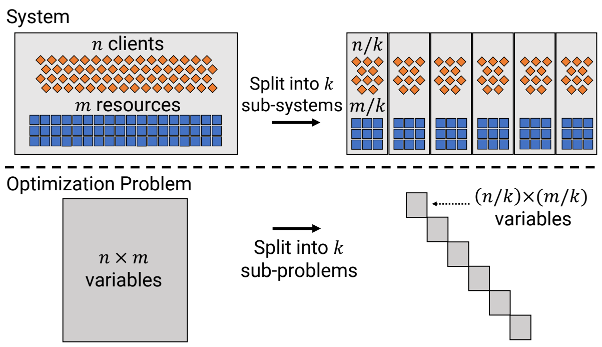


Figure 3. POP partitions the system to reduce the number of optimization problem variables. For a problem where the number of variables is the number of clients times the number of resources, dividing clients and resources evenly among k sub-problems reduces the number of variables in each sub-problem by k^2.

as shown in the top half of Figure 3. We find that this partitioning scheme is effective even when clients have attributes with skew (e.g., jobs in a shared cluster with various priority levels, or data shards in query load balancing with different loads). Low-quality allocations can also result from clients having vastly different utilities with different resources. For example, a resource could be a network link between two sites in a Wide Area Network (WAN). A commodity might *have* to use this link to send traffic between these two sites. This paper shows how client and resource splitting (§3.3) can be used to transform some of these "hard" problems into a form that is then amenable to *random partitioning*. Other broad partitioning strategies can also be used depending on problem structure (e.g., assign all "geographically close" clients and resources to the same sub-problem), but these are out of the scope of this paper. With random partitioning, the **reduce** step is cheap, as simply concatenating sub-system allocations yields a feasible allocation to the original problem.

3.3 Transformations to Granularize Problems

In some cases, it might not be possible to either return an allocation that is feasible or high quality by merely assigning each client and resource to sub-problems at random when using the POP procedure. Skewed workloads with heavy tails are common in practice [46]. As an example, consider a query load balancing problem where we try to assign shards containing various keys to compute servers: our goal is to spread load evenly amongst the available servers, which can be formulated as a mixed-integer linear program (§4.3). In such a setting, it is common for single shards to be *hot*: for example, Taylor Swift's Twitter account receives much more request traffic compared to the average Twitter user. In light of these hot shards, it might not be possible to assign shards to individual sub-problems and obtain sub-problems with input distributions similar to the original problem (and consequently leading to either an infeasible or poor-quality allocation). Similarly, in the traffic engineering problem, it is common for a small number of commodities to have large

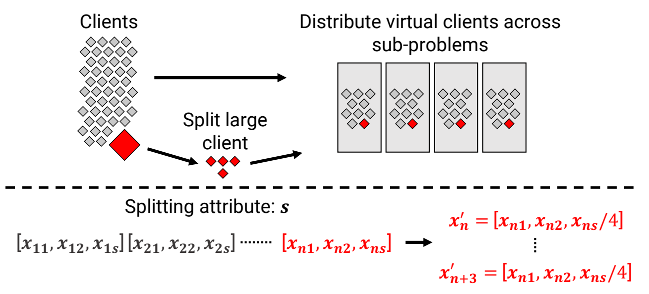

Clients
Distribute virtual clients across sub-problems
Split large client
Splitting attribute: s
$[x_{11}, x_{12}, x_{1s}]\,[x_{21}, x_{22}, x_{2s}]$ .... $[x_{n1}, x_{n2}, x_{ns}]$
$x'_n = [x_{n1}, x_{n2}, x_{ns}/4]$
$x'_{n+3} = [x_{n1}, x_{n2}, x_{ns}/4]$

Figure 4. Client splitting, where we *granularize* non-granular problems by splitting clients based on a splitting attribute s.

Algorithm 2 Client Splitting Algorithm.

```
Input: Inputs $X = [x_1, x_2, \ldots, x_n]$, splitting attributes $s$, ratio of extra virtual clients $t$ allowed.
Return: Mapping from real to virtual clients $\{x_i \rightarrow [x'_j]\}$.

Initialize queue $\leftarrow$ MAX_HEAP(), mapping $\leftarrow \{\}$.
For all $i \in \{1, 2, \ldots, n\}$, queue.PUSH($x_i.s, x_i$).

while len(queue) $\leq (1 + t) \cdot n$ do
    $x_{\max}$ = queue.POP()
    Split $x_{\max}$ by attribute $s$ into two copies $x^1_{\max}$ and $x^2_{\max}$ ($x^1_{\max}.s, x^2_{\max}.s = x_{\max}.s/2$).

    UPDATE_MAPPING($x_{\max}$, $[x^1_{\max}, x^2_{\max}]$)
    queue.PUSH($x^1_{\max}.s, x^1_{\max}$), queue.PUSH($x^2_{\max}.s, x^2_{\max}$)
end while
```

demands [5]. Multiple such commodities in a sub-problem would lead to sub-optimal total flow. To transform these into granular problems, we propose an algorithm to *split* variables for clients and resources across several sub-problems.

Client Splitting. We require the user to specify the client attribute that specifies resource demand and can be split across several sub-problems; all other attributes are copied over without change. In the load balancing example, where clients are data shards and attributes include shard load and memory size, the *splitting attribute* is the shard load. In the traffic engineering example, the splitting attribute is the commodity's traffic demand. Given this splitting attribute, we then construct a priority queue (heap) of the corresponding attribute values for all clients. Given a threshold t (t is typically a number less than 1) on the maximum number of extra *virtual* clients allowed, we pop and split variables off the queue, and then push the new variables back into the queue. Each split reduces the value of the splitting attribute of the popped variable by a factor of 2. Importantly, each split maintains the feasibility invariant: the coalesced allocation across virtual clients will still be feasible (since the total sum of splitting attribute values remains the same). By reducing the value of the splitting attribute, client splitting breaks down large clients into a collection of smaller clients with *equivalent* total demand. The runtime of this algorithm is $O(n \log n)$, where n is the number of clients, which is cheap compared to the runtime of allocation computation in each sub-problem. Algorithm 2 shows pseudocode, and the procedure is illustrated in Figure 4. Empirically, we found that most problems are granular enough for POP to work well with 0 split clients. Client splitting does not adversely impact allocation quality, but can increase runtime. The hardest problems in our experiments required $t = 0.75$. The optimal value of t is problem-specific and it is possible that users may have to dynamically adapt t to get the best performance from POP; however, in all of the considered production use-cases in our experiments, we found that small values of t that worked well for historical problem instances continue to work well on future problem instances.

Resource Splitting. If a client has to use a particular resource to make progress, POP will not work out of the box, since randomly partitioning clients and resources into sub-problems might result in a partitioning where the client is not matched with its preferred resource. In such cases, each resource can be split into k "virtual" resources (where k is the number of sub-problems). Each virtual resource has $k\times$ lower capacity, and is assigned to a different sub-problem. By ensuring that each virtual resource has lower capacity, we ensure that the final coalesced allocation is still feasible.

Client and resource splitting are not always applicable. For example, resource splitting cannot be used easily if the allocation problem's objective depends on whether a resource is used or not (e.g., an allocation problem that tries to minimize the number of resources used). Similarly, client splitting cannot be used easily for problems which take into account interactions between multiple clients sharing a resource.

The resulting allocation problem after these transformation steps can be granular; if so, we can use POP to solve it. After the `partition` step, we obtain allocations for each virtual variable in the problem. Allocations assigned to virtual variables corresponding to a single client need to be summed to obtain the final allocation. We show how this can be incorporated into the full POP procedure in Algorithm 1.

3.4 Benefits of POP

POP has several desirable properties:

- **Simplicity:** Users do not need to design new heuristics from scratch to scale up to larger problem sizes, and can reuse their original problem formulations.
- **Generality across domains and solvers:** POP can be used to accelerate allocation computations for many different types of problem formulations across domains. POP also easily integrates with different solvers.
- **Applicability to different types of objectives:** POP can be applied for a broad class of objectives, such as

total flow and maximum concurrent flow in traffic engineering. These objectives have traditionally required very different approximation algorithms [16, 25].

- **Composability:** POP can be used for any granular allocation problem in an outer loop as a simplifying step; existing heuristics or approximation algorithms can then be used to solve the resulting sub-problems.
- **Tunability:** The number of sub-problems is a knob for trading off between allocation quality and runtime.

4 Case Studies of Applying POP

In this section, we describe various resource allocation problems that are formulated as optimization problems: scheduling of jobs on clusters with possibly heterogeneous resources [32], WAN traffic engineering [5], and query load balancing [12, 42, 45]. We show the full *exact* problem formulations presented in the corresponding papers, and then explain how POP can be used to compute high-quality allocations faster. We also present some examples of problems which are not granular and out of scope for POP.

4.1 Resource Allocation for Heterogeneous Clusters

We first discuss the optimization problem formulations used in Gavel, which supports a range of complex objectives. These can be accelerated using POP since these problems are granular, i.e., meet the conditions in §2.

Gavel [32] is a cluster scheduler that assigns cluster resources to jobs while optimizing various multi-job objectives (e.g., fairness, makespan, cost). Gavel assumes that jobs can be time sliced onto the available heterogeneous resources, and decides what fractions of time each job should spend on each resource type by solving an optimization problem. Optimizing these objectives can be computationally expensive when scaled to 1000s of jobs, especially with "space sharing" (jobs execute concurrently on the same resource), which requires variables for every *pair* of runnable jobs.

Allocation problems in Gavel are expressed as optimization problems in terms of a quantity called *effective throughput*: the throughput a job observes when given a resource mix according to an allocation A, computed as:

$$\text{throughput}(\text{job } j, \text{allocation } A) = \sum_i T_{ji} \cdot A_{ji}.$$

T_{ji} is the raw throughput of job j on resource type i. In Gavel, vanilla heterogeneity-aware allocations A_{ji} are assigned to each combination of job j and GPU type i. A_{ji} represents the fraction of wall-clock time that a job j should spend on the GPU type i. We now show formulations for three objectives.

Max-Min Fairness. The Least Attained Service policy [20] tries to give each job an equal resource share of the cluster. The heterogeneity-aware version of this policy can be expressed as a max-min optimization problem over all active jobs in the cluster. We assume that each job j has fair-share weight w_j and requests z_j GPUs. Then, to take into account the impact of moving a job between GPU types, we find the max-min allocation of normalized effective throughputs:

$$\text{Maximize}_A \min_j \frac{1}{w_j} \frac{\text{throughput}(j, A)}{\text{throughput}(j, A^{\text{equal}})} \cdot z_j.$$

A^{equal} is the allocation given to job j assuming it receives equal time share on each worker type in the cluster. We also need to specify constraints to ensure that jobs and the cluster are not over-provisioned (e.g., total GPU allocation time does not exceed the total number of GPUs):

$$\begin{array}{ll} 0 \le A_{ji} \le 1 & \forall (j, i) \\ \sum_i A_{ji} \le 1 & \forall j \\ \sum_j A_{ji} \cdot z_j \le \text{num\_workers}_i & \forall i \end{array}$$

The above formulation can be extended to consider space sharing [32, 49], where multiple jobs execute concurrently on the GPU to improve GPU utilization, by only changing the way effective throughput is computed; see the Gavel paper [32] for details.

Proportional Fairness. Proportional fairness [6] tries to maximize total utilization while still maintaining some minimum level of service for each user (in this case, job). Proportional fairness for GPU cluster scheduling can be formulated as the following convex optimization problem:

$$\text{Maximize}_A \sum_j \log(\text{throughput}(j, A)).$$

Constraints are the same as before. Per-job weights and other extensions are also possible (the above objective can be interpreted as a sum of utilities, i.e., $\text{Maximize}_A \sum_i U_i(A_i)$).

Minimize Makespan. We can also minimize makespan (the time taken by a collection of jobs to complete) using a similar optimization problem framework. Let num_steps_j be the number of iterations remaining to train job j. The makespan can then be computed as the maximum of the durations of all active jobs; the duration of job j is just the ratio of the number of iterations to $\text{throughput}(j, A)$. Mathematically, this can be written as follows using the same above constraints:

$$\text{Minimize}_A \max_j \frac{\text{num\_steps}_j}{\text{throughput}(j, A)}.$$

Using POP. We can use POP on these cluster scheduling problems by partitioning the full set of jobs into job subsets, and the cluster into sub-clusters. Each sub-cluster has an equal number of resources (GPUs of each type), and jobs are partitioned randomly into the job subsets. The POP solution is feasible by construction. Since the cluster has multiple resources of each type (e.g., GPU of specific generation), the problem is granular by default, and does not require additional transformations to be made granular. Additionally, even when allowing job colocation (using space sharing), jobs can make progress colocated with many other jobs.

4.2 Traffic Engineering and Link Allocation

We next discuss optimization problem formulations that require both resource and client splitting to be solved accurately and efficiently by POP.

The problem of traffic engineering for networks determines how flows in a Wide Area Network (WAN) should be allocated fractions of links of different capacities to best satisfy a set of demands. One might consider several objectives, such as maximizing the total amount of satisfied flow, or minimizing the extent to which any link is loaded to reserve capacity for demand spikes.

Maximize Total Flow. The problem of maximizing the total flow, given a matrix of per-commodity demands D (each commodity or flow j has a demand D_j), a pre-configured set of paths P, and a list of edge capacities c_e, can be written as:

$$\text{Maximize}_A \sum_{j \in D} A_j.$$

Subject to the constraints:

$$\begin{array}{ll} A_j = \sum_p A_j^p & \forall j \in D \\ A_j \leq D_j & \forall j \in D \\ \sum_{\forall j, p \in P_j, e \in p} A_j^p \leq c_e & \forall e \in E \\ A_j^p \geq 0 & \forall p \in P, j \in D \end{array}$$

A_j^p is the flow assigned to commodity j along path p (one of the paths in P_j). The constraints ensure that the total flow through an edge does not exceed the capacity of the edge, that each commodity's flow per path is positive, and each commodity's flow does not exceed its demand.

For every commodity, the set P consists of a pre-computed set of paths between the source and target nodes [5].

Maximize Concurrent Flow. The objective only needs to be changed to:

$$\text{Maximize}_A \min_{j \in D} A_j.$$

The constraints are the same as above.

Using POP. To accelerate allocation computation using POP, we need to granularize the original problems. In particular, we use resource splitting for all traffic engineering problems: we assign the entire network (all nodes and edges) to each sub-problem (but each link with a fraction of the total capacity), and distribute commodities across sub-problems. We do not shard the network itself (i.e., assign each link to a single sub-problem only) since traffic can flow between any pair of nodes and the difference in utility for any commodity when using a fraction of the available links in the network is high (links between specific sites may *need* to be used to sustain sufficiently high flow). By assigning each sub-problem a link with a fraction of the total capacity, we ensure that the final allocation from POP is feasible. For specific problems with large commodities, we also use client splitting.

4.3 Query Load Balancing

Systems like Accordion [42], E-Store [45], and Kairos [12] need to determine how to place data items in a distributed store to spread load across available servers.

We consider the problem of load balancing data shards (collections of data items). This is similar to the single-tier load balancer in E-Store, but acting on collections of data items instead of individual tuples. The objective is to minimize shard movement across servers as load changes, while constraining the load on each server to be within a tolerance ϵ of average system load L. Each shard i has load l_i and memory footprint f_i. Each server j has a memory capacity of memory_j that restricts the number of shards it can host. The initial placement of shards is given by a matrix T, where $T_{ij} = 1$ if partition i is on server j. A is a shard-to-server map, where A_{ij} is the fraction of queries on partition i served by j, and $A'_{ij} = 1$ if $A_{ij} > 0$, 0 otherwise. Finding the balanced shard-to-server map that minimizes data movement can then be formulated as a mixed-integer linear program:

$$\text{Minimize}_A \sum_i \sum_j (1 - T_{ij}) A'_{ij} f_i.$$

Subject to the constraints:

$$\begin{array}{ll} L - \epsilon \leq \sum_i A_{ij} l_i \leq L + \epsilon & \forall j \\ \sum_j A_{ij} = 1 & \forall i \\ \sum_i A'_{ij} f_i \leq \text{memory}_j & \forall j \\ A_{ij} < A'_{ij} \leq A_{ij} + 1 & \forall (i, j) \end{array}$$

Using POP. The load balancing problem can be accelerated using POP by dividing the shard set and server cluster into shard subsets and server sub-clusters, while ensuring that each shard subset has the same total load.

4.4 When is POP Not Applicable?

Although POP can be used on a number of different resource allocation problems, it cannot be used for *all* possible problem formulations. Here, we present a few examples of resource allocation problems where POP with random partitioning *cannot* be used.

Capacitated Facility Location. The capacitated facility location problem tries to minimize the cost of satisfying users' demand given a set of processing facilities. Each facility has a processing capacity, and also a "leasing cost" if used at all (if a facility is not processing any demand, it has a leasing cost of 0). The cost of processing some demand by a facility is proportional to the distance of the facility from the user. Problems where a user is only close to a single facility are not amenable to POP and violate **condition 3** in the definition of granularity: partitionings of the problem where the user is not placed into the same sub-problem with the facility closest to them would lead to a low-quality allocation. Additionally, resource splitting cannot be used to make the problem granular, since the objective explicitly takes into account whether facilities are used or not, and creating multiple variables for a single <client, resource> pair would require additional constraints *across* sub-problems. More generally, resource allocation problems where clients prefer one resource over all other available resources by a large amount are a poor fit for POP unless resource splitting can be used.

Traffic Engineering. A variant of the traffic engineering problem from §4.2 could include hard constraints like "flows A and B should / should not use the same link". This violates **condition 4** in the definition of granularity. Randomly partitioning clients and resources into sub-problems would not work all the time (e.g., random partitioning could drop flows A and B into different sub-problems when flows A and B need to use the same link); smarter partitioning algorithms can mitigate this by considering affinity between flows, but supporting these is left to future work.

Global Rescheduling with Plan-Ahead. TetriSched [47] is a scheduler that can take into account upcoming resource reservations when deciding how to allocate resources to jobs. TetriSched allows preferences to be specified declaratively (e.g., a job comes in at a specific start time and needs to be completed by a specific end time). These preferences are then compiled into a mixed-integer linear program (MILP). These MILPs can be accelerated using POP by dividing the jobs and resources into job and resource subsets, and solving each sub-problem independently. However, TetriSched also supports combinatorial constraints, such as "a particular set of k jobs must use the same resource", which cannot be supported by POP without smarter partitioning algorithms.

5 Analysis

The effectiveness of POP is directly tied to how clients and resources are partitioned across sub-problems. In this section, we consider a simple resource allocation problem and prove that the probability of a large optimality gap with the POP procedure and random partitioning is low, discuss how POP relates to *primal decomposition* (a technique used in convex optimization to decompose certain types of optimization problems), and also note the expected runtime benefits.

5.1 Theoretical Analysis for a Simple Problem

In settings with large numbers of clients, POP with random partitioning works well. In this section, we consider a simplified allocation problem and compute an upper bound on the probability that POP (using k sub-problems) with random partitioning results in a low-quality allocation.

The allocation problem we consider assigns servers to jobs. We assume that the problem has the following properties:

- n jobs and servers. Each job is allocated a single server.
- r distinct server types (equal number of each type).

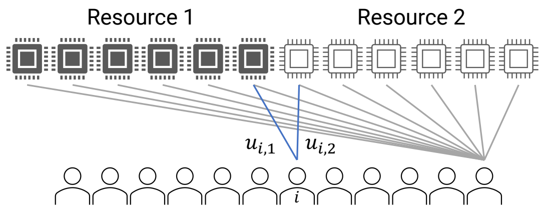


Figure 5. Simple partitioning problem where jobs are assigned servers (or resources). Each job i derives utility $u_{i,1}$ from resource 1 and $u_{i,2}$ from resource 2.

- Job i has utility $u_{i,s}$ on resource type s.
- The largest difference in utility for any job across any two servers is u_{maxgap}.

A job is "type-s" if it achieves highest utility on a type-s server. With two server types, we have type-1 and type-2 jobs (shown in Figure 5).

The objective of this problem is to maximize the overall utility of the allocation, defined as the sum of every job's utility on its assigned server.

Now, if we use POP to solve this problem, we would equally partition servers of each type into sub-clusters, randomly assign jobs to sub-clusters, and then solve assignment problems separately for each sub-cluster. We wish to answer the following questions in this regime:

1. What is the optimality gap of the solution using the POP procedure (with respect to the optimal solution for the full problem)?
2. How do the values of n, r, u_{maxgap}, and k affect this optimality gap?

One way to quantify the optimality gap is to count the number of "misplaced" jobs in each sub-problem (e.g., type-1 jobs that are not assigned "resource 1" because there were too many other type-1 jobs in the relevant sub-problem). Define $q_{s,t}$ to be the number of type-s resources that are misplaced in sub-problem t. The distance from optimal utility, i.e., optimality gap, is bounded by the product of this number and u_{maxgap} added across all resource types and sub-problems:

$$\text{Optimality gap} \leq \sum_{s=1}^{r} \sum_{t=1}^{k} q_{s,t} u_{\text{maxgap}} \tag{1}$$

We note that this is a loose bound for the gap, since jobs with large resource utility gaps would be allocated their optimal resource even within a sub-problem.

To quantify the performance gap between POP and optimal solutions, we now need a sense of how big $q_{s,t}$ can be in practice. We walk through the full derivation of a bound on the probability that the optimality gap exceeds a given value in the Appendix, but briefly sketch it here. The random assignment of all type-s jobs to sub-problems can be interpreted as Bernoulli trials where the probability that any given type-r job is placed in a given sub-problem is

$1/k$. We then use a classical Chernoff bound [31] to compute the probability that each $q_{s,t}$ exceeds a fraction δ of its expected value (n/rk). We can combine these across all job types and sub-problems using the union bound to find an upper limit on the probability that the total number of misplaced jobs exceeds δn. This allows us to bound the distance of a randomly-partitioned POP allocation from optimal utility by $\delta u_{\text{maxgap}} n$:

$$\Pr\left[U(\Gamma^*) - U(\Gamma^{\text{POP}}) \geq \delta u_{\text{maxgap}} n\right] \leq rk \exp\left(\frac{-\delta^2 n}{(2+\delta)rk}\right) \tag{2}$$

where Γ^* is an optimal allocation, Γ^{POP} is the allocation returned by the POP procedure, and $U() : \Gamma \rightarrow u$ is a function that maps an allocation Γ to a scalar value (the utility).

Equation 2 defines the relationship between the problem parameters (n, r, u_{maxgap} and k) and the probability that the optimality gap exceeds a given fraction δ of the worst-case gap if every job is allocated its worst resource ($u_{\text{maxgap}} n$). Concretely, the probability decays exponentially with n; as the problem gets larger, the probability of having a large optimality gap becomes very small. The probability also decays exponentially with δ^2. On the other hand, the probability of a large optimality gap increases as r, k, and u_{maxgap} increase; this is to be expected, as having many sub-problems and many resource types increases problem heterogeneity and makes it more likely for a random partitioning to lead to misplaced jobs and a lower-quality allocation.

To put this bound into perspective, consider a large cluster with 1 million jobs, $k = 10$ sub-problems, and $r = 4$ resource types of equal amounts ($n/rk = 25,000$); the probability that more than 3% of jobs are not allocated their optimal resource is upper bounded by 0.000614.

To summarize, the bound given in Equation 2 for a simple allocation problem gives insight as to why POP works well empirically for more complex granular resource allocation problems like those described in §4.

5.2 Relationship to Primal Decomposition

For many problems, such as when the objective function is separable and convex (that is, the objective can be expressed in the form "Maximize $U(A) = \sum_i U_i(A_i)$" with per-job utility functions U_i), POP can be interpreted as the first iteration of primal decomposition, a well-known method from convex optimization [10]. Primal decomposition is an iterative technique; for a resource allocation problem, it works by decomposing the large problem into several smaller allocation problems, each with a subset of clients and resources. In each iteration, every sub-problem is solved individually, and then the dual variables of each sub-problem are used to determine how to shift resources between the sub-problems; those found to be relatively resource-starved are given more resources from other sub-problems for the next iteration.

Like many other techniques from the optimization literature, primal decomposition works for a restricted set of problems, namely those with separable objectives and certain types of constraints (see Boyd et al. [10]). These restrictions come into effect during the resource-shifting phase prior to subsequent iterations. For a "well-partitioned" problem with a separable objective (i.e., each sub-problem has sufficient resources), one iteration of primal decomposition is often sufficient and resource shifting is not required [10]. Primal decomposition and POP are thus equivalent for these problems, explaining why POP can produce a high-quality allocation efficiently. However, this explanation does not apply to other problems where primal decomposition cannot be used (e.g., non-convex problems, such as the MILP used in the load balancing problem from §4.3), even though we found POP to still be effective in such regimes.

5.3 Expected Runtime Benefits

We can estimate the runtime benefits of POP when used with linear programs. Solvers for linear programs have worst-case time complexity of $O(f(n,m)^a)$ ($a \approx 2.373$ [11] in the worst case) where $f(n,m)$ is the number of variables (n clients and m resources) in the problem. If $f(n,m) = n \cdot m$ and both clients and resources are partitioned across k sub-problems, each sub-problem will have $k^2\times$ fewer variables, as illustrated in Figure 3. The asymptotic runtime savings are then proportional to k^{2a-1} if each sub-problem is solved serially, and proportional to k^{2a} if solved in parallel, assuming a cheap `reduce` step. Some problems have an even larger potential for runtime reduction. For example, if the allocation considers interactions between two jobs on the same resource, then the problem would have $n^2 m$ variables, and using POP would lead to a larger runtime speedup (proportional to k^{3a-1} if each sub-problem is solved serially, and proportional to k^{3a} if solved in parallel).

6 Implementation

POP is easy to implement on top of a number of existing solvers for a variety of different granular allocation problems. The main method that needs to be implemented is `partition`, which given a collection of clients and resources, assigns them to sub-problems. The subsequent `map` step then involves calling the existing solver routine for the already-written problem formulation on the smaller sub-problem. The `reduce` step is similarly simple, and involves concatenating the allocations obtained from each of the sub-problems and summing allocations across virtual clients and resources (when using client and resource splitting).

We implemented POP on top of a number of different solvers (MOSEK using `cvxpy` [7, 15], Gurobi [21], and a custom solver [6] that uses PyTorch [37]) for problems across diverse domains, in < 20 lines of code in each case. We implemented client splitting in about 100 lines of Python code.

7 Evaluation

In this section, we seek to answer the following questions:

1. What is the effect of POP on allocation quality and execution time on granular allocation problems? How does it compare to relevant heuristics?
2. Does POP work across a range of solvers and types of optimization problems?
3. How effective are POP's client and resource splitting optimizations in generating high-quality allocations?
4. How does random partitioning compare to other more sophisticated problem partitioning strategies?

We evaluate POP on problems from three domains:

1. **GPU cluster scheduling**, where we apply POP to solve the optimization problems used in Gavel (§4.1), and compare with the greedy Gandiva policy [49].
2. **Traffic engineering** across Wide Area Networks, where we apply POP to solve the problem formulations in §4.2, and compare to CSPF and NCFlow [5].
3. **Shard load balancing** in distributed storage systems, where we apply POP on the problem formulation in §4.3, and compare to a heuristic from E-Store [45].

Where relevant, we integrate POP into systems such as Gavel [32] to measure the end-to-end impact of POP on application performance. Our results span three different cluster scheduling policies (max-min fairness, minimize makespan, and proportional fairness), two traffic engineering policies (maximize total flow, and maximize concurrent flow), and one load balancing policy (minimize number of shard transfers as load changes).

We first present end-to-end experiments, then present some microbenchmarks that examine the impact of various algorithmic contributions in POP.

7.1 End-to-End Results

We first demonstrate POP's end-to-end effectiveness on various problems. We compare to approaches based on allocation quality, and time needed to compute the allocation; the runtime for POP includes the runtime for solving the optimization problems for sub-problems. In all of our experiments, "Exact sol." is the original unpartitioned problem formulation and solver used by the reference system (e.g., Gavel for cluster scheduling). We believe this is a fair baseline since it represents what people use today if using optimization problem formulations for resource allocation. We use the same evaluation methodology as related work. The total number of threads given to solvers for our baselines and POP are the *same*. If k sub-problems are solved in parallel when using POP, each sub-problem uses $1/k$ of the number of threads. We also present heuristics where relevant. Unfortunately, not every problem has a state-of-the-art heuristic. For example, it is not clear how to use a heuristic to solve for an approximate proportionally-fair allocation. We explicitly note when we use client or resource splitting.

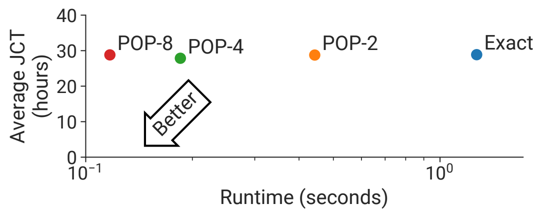


Figure 6. Results for the max-min fairness policy (with space sharing) for cluster scheduling for the formulation shown in §4.1 ("Exact sol.") and its POP variants. POP-k uses k sub-problems.

7.1.1 Cluster Scheduling

We used POP to accelerate various cluster scheduling policies supported by Gavel [32]. We then used these POP-ped policies in Gavel's full simulator[1] to measure the impact of POP on end-to-end metrics of interest, like average job completion time and makespan for real traces. The traces and methodology used are identical to those used in Gavel.

Max-Min Fairness. We show the trade-off between runtime and allocation quality for the max-min fairness policy with space sharing on a large problem (2048^2 job pairs on a 1536-GPU cluster) in Figure 2 (in the introduction). POP leads to an extremely small change in the average effective throughputs across all jobs (< 1%), with a 22.7× improvement in runtime. Gandiva [49], on the other hand, uses a heuristic to assign resources to job pairs, resulting in 1.9× worse allocation quality.

We unfortunately could not run end-to-end simulations for such large problem sizes: the simulation involves running thousands of allocation problems, since an allocation problem needs to be solved every time a new job arrives at the cluster or an old job completes. This would take months to run at scale by virtue of the number of problems that need to be solved and the time taken for each problem. Instead, we show full simulation results on more moderate problem sizes. These experiments involve dynamic changes: the full simulation involves new jobs coming in and old jobs completing, and consequently the set of jobs is not static.

We ran experiments with 96 GPUs (32 V100, P100, and K80 GPUs). The original heterogeneity-aware Least Attained Service policy without space sharing has a small number of variables (on the order of hundreds). Even on such smaller problem sizes, the quality of allocation with POP is high, with only up to a 5% drop in average JCT (not pictured).

Figure 6 shows the average JCT of the original Least Attained Service policy from §4.1, with space sharing, along with three POP-ified versions using 2, 4, and 8 sub-problems. With space sharing, the number of variables scales quadratically with the number of jobs: this leads to a performance

[1]The Gavel paper [32] shows that its simulator demonstrates performance very similar to behavior on the physical cluster.

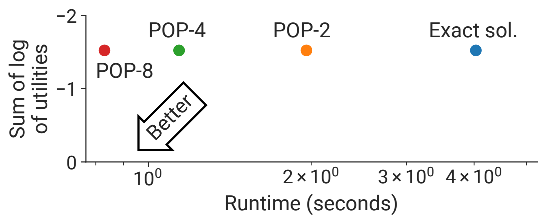


Figure 7. Results for the proportional fairness policy for cluster scheduling for the formulation shown in §4.1 ("Exact sol.") and its POP variants. POP-k uses k sub-problems.

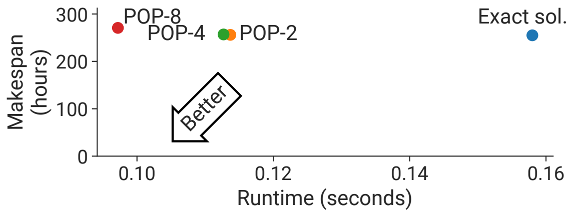


Figure 8. Results for the minimize makespan policy for cluster scheduling for the formulation shown in §4.1 ("Exact sol.") and its POP variants. POP-k uses k sub-problems.

speedup of 11× with $k = 8$ compared to the full problem formulation, and similar average JCT.

We see similar behavior for max-min fairness policies when clients have more attributes (e.g., different priority levels). Average JCTs are almost identical when jobs request multiple GPUs, and increase by 5% for high-priority jobs in workloads containing a mix of low- and high-priority jobs, using the Gavel simulator as before.

Proportional Fairness. We ran a simple experiment with the proportional fairness policy with 10^6 jobs and a similar number of resources. Figure 7 shows POP combined with a proportional fairness policy. This allocation problem is a general convex optimization problem (not a linear program), with a sum-of-log objective. For this problem, we implement POP on top of a custom solver [6] that runs an order of magnitude faster than commercial solvers for this particular problem formulation. We see strong scaling performance as we increase the number of sub-problems (4.9× reduction in runtime with 8 sub-problems), with an extremely small optimality gap (7×10^{-5}).

Minimize Makespan. Figure 8 shows the makespan of variants of the "minimize makespan" policy. This policy again is a simple linear program with number of variables linear in the number of jobs and resource types. Consequently, the runtime improvements are lower (1.6×), but the end-to-end makespan over the trace is nearly identical.

7.1.2 Traffic Engineering

We tested POP on several large networks (shown in Table 1) from the Topology Zoo repository [26], with similar results.

Topology	# Nodes	# Edges
Kdl	754	1790
Cogentco	197	486
UsCarrier	158	378
Colt	153	354
GtsCe	149	386
TataNld	145	372
DialtelecomCz	138	302
Deltacom	113	322

Table 1. The WAN topologies used to benchmark POP for traffic engineering problems, obtained from Internet Topology Zoo [26].

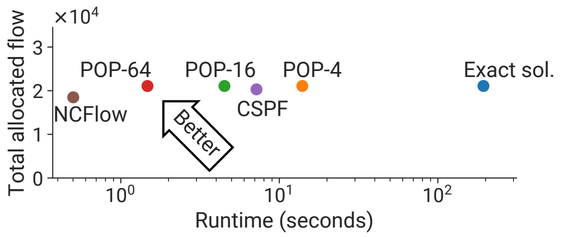


Figure 9. Results for the max-flow problem for traffic engineering for a single topology and traffic matrix. The scatterplot shows runtimes and total allocated flow for the formulation shown in §4.2 ("Exact sol.") and its POP variants, as well as CSPF and NCFlow.

For each topology, we benchmarked POP on sets of synthetic traffic matrices, which were generated using several traffic models: Gravity [8, 41], Uniform, Bimodal [8], and Poisson. These traffic matrices were previously used in NCFlow[2] [5]. Poisson represents a skewed workload, where a small percentage of commodities dominate the network demand. For this workload, we use the client-splitting algorithm from §3.3 to improve allocation quality. We do not use the client-splitting algorithm for the other traffic matrices.

Total Flow. Figure 9 shows the trade-off between runtime and allocated flow on the Kentucky Data Link network (Kdl in Table 1), which has 754 nodes and 1790 edges spanning the Eastern half of continental USA. We instantiated over 5×10^5 demands to up to 4 paths in the network. The flow allocated by POP is within 1.5% of optimal when using 64 sub-problems, yet 100× faster than the original problem. We also compare favourably to the Constrained Shortest Path First (CSPF) heuristic [17] and the recently-published NCFlow [5]. Note that NCFlow is *not* a heuristic, but a state-of-the-art approach that uses a problem decomposition technique explicitly tuned for the max-flow problem.

Figure 10 shows the improvement in allocation quality and runtime compared to the original LP formulation presented in §4 with POP using 16 sub-problems. Each point in the scatterplot represents a different topology and traffic matrix. We see larger speedups for the larger Kdl topology. We used client splitting with a threshold (t) of 0.75 for the

[2]The full set of traffic matrices can be found here: https://github.com/netcontract/ncflow.

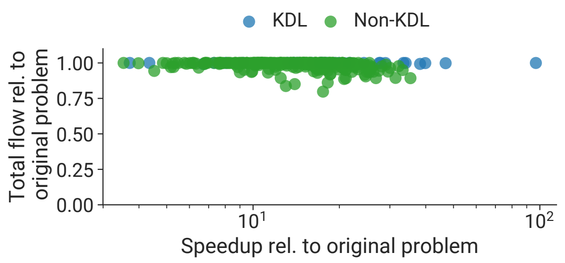


Figure 10. Results for the max-flow problem for traffic engineering for multiple topologies and traffic matrices. The scatterplot shows runtimes and allocated total flow for POP-16 across 275 experiments, separated by large (kdl) and small (non-kdl) topologies.

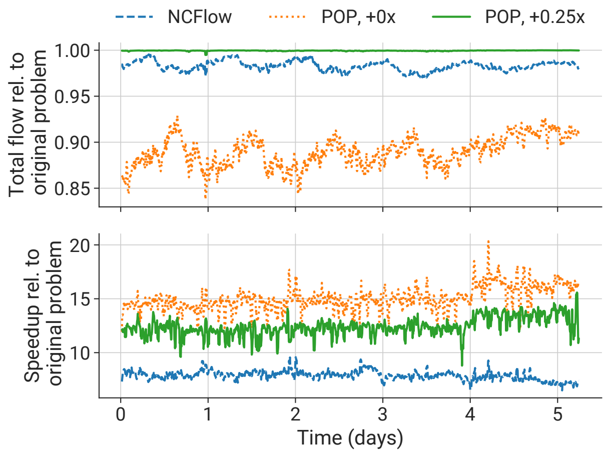


Figure 11. Allocated flow and speedup relative to original problem on a 5-day sequence of real-world traffic matrices from a private WAN with 100s of nodes and edges. With client splitting ($t = 0.25$), POP allocates >99% of the total flow with a 12.5× median speedup.

Poisson traffic matrices (where some commodities have large demands), and no client splitting for the other traffic models, which were granular out of the box. As stated before, resource splitting was used for all traffic matrices to ensure that each sub-problem has all links (but with lower capacity).

We also ran experiments with a sequence of *real-world* traffic traces collected on a private industrial WAN with hundreds of nodes and edges. Figure 11 plots the moving average (over 5 windows) of the total flow and speedup relative to the original problem for NCFlow, POP with no client splitting, and POP with $t = 0.25$ client splitting. Without client splitting, POP achieves significant speedups (15× in the median case) compared to the original problem, but allocates 89.1% of the total flow in the median case. However, POP with client splitting nearly matches the total flow allocated in the original problem (99.9% in the median case), while still achieving a median 12.5× speedup.

Maximum Concurrent Flow. Similarly, we benchmarked POP on the maximum concurrent flow objective using the

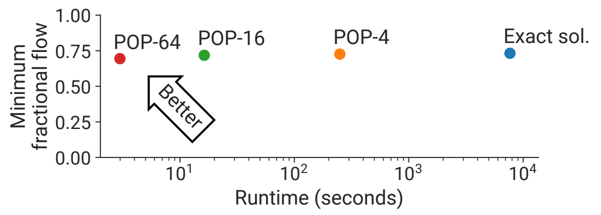


Figure 12. Results for the maximum concurrent flow problem for traffic engineering for a single topology and traffic matrix. The scatterplot shows runtimes and minimum fractional flow for the formulation shown in §4.2 ("Exact sol.") and its POP variants.

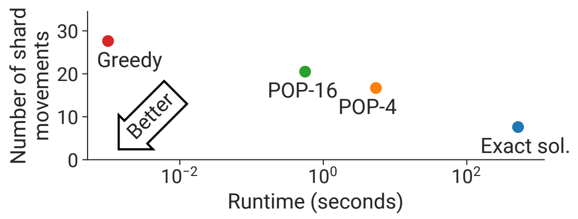


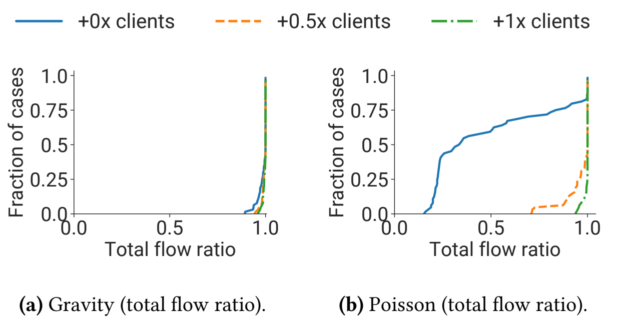
Figure 13. Results for the minimize shard movement policy for load balancing for the formulation shown in §4.3 ("Exact sol.") and its POP variants. We also compare to a greedy heuristic ("Greedy"). POP-k uses k sub-problems.

same set of topologies and traffic matrices. Figure 12 shows the trade-off between runtime and minimum fractional flow on the Kdl topology, using the same traffic matrix in Figure 9. The objective value realized by POP is again within 1.5% of optimal when using 64 sub-problems, yet 1000× faster than the original problem. As before, we use client splitting with a threshold of 75% for the Poisson traffic matrices, and no client splitting for the other traffic matrices. Resource splitting is used for all traffic matrices.

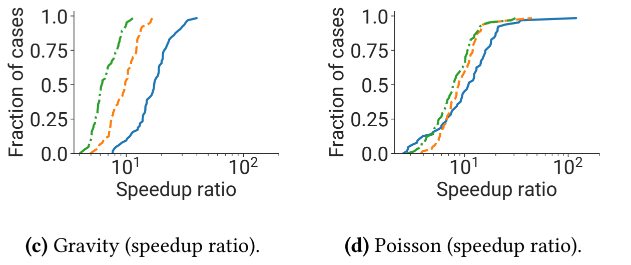
7.1.3 Load Balancing

In Figure 13, we evaluate POP on a load balancing problem. In the problem, we have 1024 shards of data each assigned to exactly one of 64 servers. Each round, we receive the query load of each shard and compute a new assignment of shards to servers such that each server has approximately (within 5%) the same amount of load across its shards but the number of shard movements is minimized. We examine the performance of POP with various numbers of sub-problems and compare it to the original optimization problem (§4.3) and a greedy heuristic algorithm from E-Store [45]. For each system, we run 100 rounds of the problem. In each round, we generating a new load distribution and rerun the load balancing algorithm. We report the average runtime and number of shard movements across these rounds.

We find that POP improves the runtime over the original problem by two orders of magnitude, while outperforming the greedy heuristic. The exponential scaling of MILP solvers restricted us to smaller problem sizes for the purpose of comparing against the optimal solution. Since shard movements



(a) Gravity (total flow ratio). **(b)** Poisson (total flow ratio).



(c) Gravity (speedup ratio). **(d)** Poisson (speedup ratio).

Figure 14. Comparison of POP-16 relative to original problem for the max-flow objective in traffic engineering, across different levels of additional split clients (0× to 1×) and traffic matrices from two traffic models: Gravity and Poisson (which is skewed).

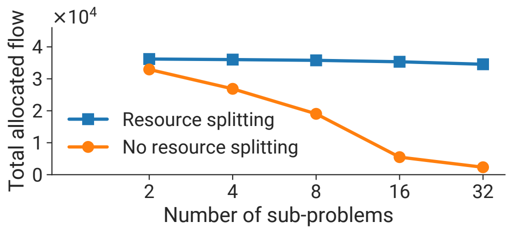


Figure 15. Effect of resource splitting on total flow as the number of sub-problems (k) is adjusted for a TE problem with a max-flow objective (Cogentco topology and Gravity traffic matrix).

are stateful (previous round's solution is initial state for current round), we added an extra step to re-balance the aggregate load in the relatively small sub-problems, requiring a few extra shard movements. As k increases, the number of sub-problems and thus the number of these movements also increases, which is why POP-k does worse as k increases. This becomes less of an issue for larger problem sizes where random allocations are likely to remain balanced.

7.2 Effectiveness of Client and Resource Splitting

Figure 14 shows the effect of client splitting on total flow and runtime when using POP with 16 sub-problems, on a traffic engineering problem with "large" clients (Poisson traffic model) as well as a more typical set of clients (Gravity traffic model) and a max-flow objective. The figure shows separate cumulative distributions of approximately 100 different experiments for each traffic model and client splitting threshold (t in Algorithm 1).

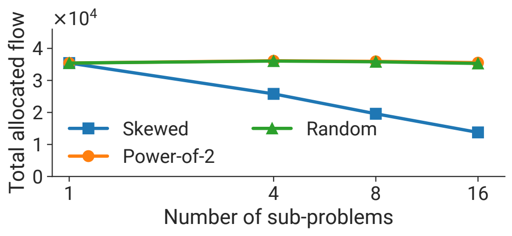


Figure 16. Performance comparison of various partitioning algorithms for the max-flow objective in traffic engineering.

We see that with skewed traffic (Poisson traffic model) and no client splitting, the total flow is typically far from optimal. Client splitting drastically increases the median relative total flow from 0.2 to near 1.0 for these problems, at the cost of some runtime overhead (due to an increase in the number of variables). In contrast, the problems with Gravity traffic get near-optimal allocated flow without client splitting.

Figure 15 shows the effect of resource splitting on total flow when using POP with various numbers of sub-problems (k) on the Cogentco topology and Gravity traffic model. Here, we split the capacity of each unique resource (link between two sites) across every sub-problem when using resource splitting. We compare this to the regular POP procedure: partitioning the network into k disjoint networks, with every link appearing in a single sub-problem. We see that total flow remains roughly the same with high k when using resource splitting. On the other hand, without resource splitting, the flow is up to 15× lower for high k. This result highlights the importance of resource splitting for problems with resources that *have* to be used by certain clients for high utility.

7.3 Alternatives to Random Partitioning

We implemented several algorithms to partition clients into sub-problems, and compared them to random. Among these is a power-of-2 partitioning algorithm that tries to assign each client sequentially to one of two randomly-chosen sub-problems using "distributional similarity" to the original problem as the metric. We also implement a skewed partitioning algorithm that deliberately creates skew among sub-problems to show the impact of bad partitions. Figure 16 shows the impact of these partitioning algorithms on the quality of allocation returned by POP on a traffic engineering problem. We see that random performs about as well as the more sophisticated power-of-2 partitioning, while skewed partitions have poor performance (skewed causes link congestion around certain nodes in the WAN).

8 Related Work and Discussion

In this section, we discuss other systems that use optimization problems to allocate resources. We also comment on general efforts to accelerate solving large optimization problems and how POP fits into this body of work.

Optimization Problems in Systems. A number of systems besides the ones discussed in §4 use optimization problem formulations to solve resource allocation problems.

TetriSched [47] is a cluster scheduler that is able to leverage runtime predictions and deadline information (provided as input to the system) to make smarter near-term decisions on how jobs should be allocated resources, while also providing room for uncertainty from unknown future job submissions. Preferences in resource space-time can be expressed in a new DSL called STRL; these are then compiled down to a mixed-integer linear program (MILP) whose solution describes when and how jobs should be executed.

RAS [33] is a capacity reservation system that manages the allocation of servers to clients within a datacenter region, while taking into account failures, resource heterogeneity, and maintenance schedules. RAS formulates problems as MILPs which are solved hourly.

DCM [44] makes it easier to implement various cluster management policies (e.g., ensure containers have enough of a particular resource, or two containers are not placed in the same rack) by having users specify cluster manager behavior declaratively through SQL queries written over cluster state maintained in a relational database. Similar to TetriSched, these queries are then compiled down to an optimization problem that can be solved by constraint solvers, such as CP-SAT [1]. DCM supports affinity and anti-affinity constraints.

Quincy [24] and Firmament [19] are centralized datacenter schedulers that use efficient min-cost max-flow (MCMF) based optimization to scale up to large clusters.

NCFlow [5] is a technique that solves the multi-commodity maximum flow problem ("total flow" in §4.2). NCFlow divides a topology into geographic clusters, and then solves a set of smaller-complexity flow problems; this yields faster runtimes and fewer forwarding entries in the WAN topology, at the cost of a smaller total flow. POP compares favorably to NCFlow, both in terms of solver runtime and total flow (see Figure 9). While POP does not offer any reduction in forwarding entries, it is more general in the objectives it can support; NCFlow only supports the max-flow objective. For the max-flow objective, POP and NCFlow can be used together to reduce solver runtime and the number of forwarding entries.

Random Partitioning in Computer Systems. Random assignment has seen success in other important systems problems as well. For example, in data center networking [43], random graph topologies work surprisingly well compared to commonly-used structured topologies such as FAT-trees. In load-balancing algorithms [30], assigning jobs to the least-loaded of just two randomly selected servers in a cluster can drastically reduce the probability of overloading a server.

Approximation Algorithms. FPTAS algorithms [4] return results with a guaranteed approximation ratio and run in polynomial time over this approximation factor. Proving an approximation ratio with POP is hard since we apply POP to many different problems with various structures, as opposed to designing a problem-specific approximation algorithm.

More Efficient Solving. The optimization community has developed various methods for scaling optimization solvers to handle large problems. Fundamentally, these approaches rely strictly on identifying and then exploiting certain mathematical structures (if they exist) within the problem to extract parallelism; they make no domain-aware assumptions about the underlying problem. For example, Benders' decomposition [18, 39] only applies to problems that exhibit a block-diagonal structure; ADMM [9, 36] has been applied to select classes of convex problems, and Dantzig-Wolfe decomposition [13], while more broadly applicable, offers no speedup guarantee. This poses a significant limitation when applying these methods to real-world systems, which often do not meet their criteria or would need mathematical analysis to determine if this structure exists.

As mentioned in §5.2, POP can be interpreted as the first iteration of primal decomposition for optimization problems with separable objectives and certain types of coupled constraints [10]. By randomly partitioning large numbers of clients and equally apportioning resources into sub-problems, we found that it is possible to obtain high-quality solutions with a single iteration for a broader set of allocation problem formulations, including MILPs.

9 Conclusion

In this paper, we showed how a number of resource allocation problems in computer systems are *granular* and proposed an efficient new method to solve them. Such granular allocation problems can be partitioned into more tractable sub-problems by randomly assigning clients and resources. Our technique, POP, achieves strong results across a variety of tasks, including cluster scheduling, traffic engineering, and load balancing, with runtime improvements of up to 100× with small optimality gap, and outperforms greedy ad-hoc heuristics. We hope this work motivates using POP as a simple pre-solving step when solving optimization problems that arise in computer systems.

Acknowledgements

We thank our shepherd, Malte Schwarzkopf, the anonymous SOSP reviewers, Keshav Santhanam, Kostis Kaffes, Akshay Narayan, Shoumik Palkar, and Deepti Raghavan for their feedback that improved this work. We are also grateful to production teams at Microsoft for facilitating access to datasets. This research was supported in part by affiliate members and other supporters of the Stanford DAWN project—Ant Financial, Facebook, Google, and VMware—as well as Toyota Research Institute, Cisco, SAP, and the NSF under GRF grant DGE-1656518 and CAREER grant CNS-1651570. Any opinions and conclusions expressed in this material are those of the authors and do not reflect the views of the NSF.

References

[1] Google OR-Tools. https://developers.google.com/optimization.

[2] Kubernetes. https://github.com/kubernetes/kubernetes.

[3] OpenShift. https.//openshift.com.

[4] Polynomial-Time Approximation Scheme. https://en.wikipedia.org/wiki/Polynomial-time_approximation_scheme.

[5] Firas Abuzaid, Srikanth Kandula, Behnaz Arzani, Ishai Menache, Matei Zaharia, and Peter Bailis. Contracting Wide-area Network Topologies to Solve Flow Problems Quickly. In *18th USENIX Symposium on Networked Systems Design and Implementation (NSDI 21)*, 2021.

[6] Akshay Agrawal, Stephen Boyd, Deepak Narayanan, Fiodar Kazhamiaka, and Matei Zaharia. Allocation of Fungible Resources via a Fast, Scalable Price Discovery Method. *arXiv preprint arXiv:2104.00282*, 2021.

[7] Akshay Agrawal, Robin Verschueren, Steven Diamond, and Stephen Boyd. A Rewriting System for Convex Optimization Problems. *Journal of Control and Decision*, 5(1):42–60, 2018.

[8] David Applegate and Edith Cohen. Making Intra-Domain Routing Robust to Changing and Uncertain Traffic Demands. In *SIGCOMM*, 2003.

[9] Stephen Boyd, Neal Parikh, and Eric Chu. Distributed Optimization and Statistical Learning via the Alternating Direction Method of Multipliers. *Foundations and Trends in Machine Learning*, pages 1–122, 2011.

[10] Stephen Boyd, Lin Xiao, Almir Mutapcic, and Jacob Mattingley. Notes on Decomposition Methods. *Notes for EE364B, Stanford University*, 635:1–36, 2007.

[11] Michael B Cohen, Yin Tat Lee, and Zhao Song. Solving linear programs in the current matrix multiplication time. *Journal of the ACM (JACM)*, 68(1):1–39, 2021.

[12] Carlo Curino, Evan PC Jones, Samuel Madden, and Hari Balakrishnan. Workload-Aware Database Monitoring and Consolidation. In *Proceedings of the 2011 ACM SIGMOD International Conference on Management of data*, pages 313–324, 2011.

[13] George B Dantzig and Philip Wolfe. Decomposition Principle for Linear Programs. *Operations Research*, 8(1):101–111, 1960.

[14] Jeffrey Dean and Sanjay Ghemawat. MapReduce: Simplified Data Processing on Large Clusters. *Communications of the ACM*, 51(1):107–113, 2008.

[15] Steven Diamond and Stephen Boyd. CVXPY: A Python-Embedded Modeling Language for Convex Optimization. *The Journal of Machine Learning Research*, 17(1):2909–2913, 2016.

[16] Lisa K Fleischer. Approximating Fractional Multicommodity Flow Independent of the Number of Commodities. *SIAM Journal on Discrete Mathematics*, 13(4):505–520, 2000.

[17] Bernard Fortz, Jennifer Rexford, and Mikkel Thorup. Traffic Engineering with Traditional IP Routing Protocols. *IEEE Communications Magazine*, 40(10):118–124, 2002.

[18] Arthur M Geoffrion. Generalized Bender's Decomposition. *Journal of optimization theory and applications*, 10(4):237–260, 1972.

[19] Ionel Gog, Malte Schwarzkopf, Adam Gleave, Robert NM Watson, and Steven Hand. Firmament: Fast, Centralized Cluster Scheduling at Scale. In *12th USENIX Symposium on Operating Systems Design and Implementation (OSDI 16)*, pages 99–115, 2016.

[20] Juncheng Gu, Mosharaf Chowdhury, Kang G Shin, Yibo Zhu, Myeongjae Jeon, Junjie Qian, Hongqiang Liu, and Chuanxiong Guo. Tiresias: A GPU Cluster Manager for Distributed Deep Learning. In *16th USENIX Symposium on Networked Systems Design and Implementation (NSDI 19)*, pages 485–500, 2019.

[21] Zonghao Gu, Edward Rothberg, and Robert Bixby. Gurobi Optimizer Reference Manual, version 5.0. *Gurobi Optimization Inc., Houston, USA*, 2012.

[22] Ajay Gulati, Anne Holler, Minwen Ji, Ganesha Shanmuganathan, Carl Waldspurger, and Xiaoyun Zhu. VMware Distributed Resource Management: Design, Implementation, and Lessons Learned. *VMware Technical Journal*, 1(1):45–64, 2012.

[23] Chi-Yao Hong, Subhasree Mandal, Mohammad Al-Fares, Min Zhu, Richard Alimi, Chandan Bhagat, Sourabh Jain, Jay Kaimal, Shiyu Liang, Kirill Mendelev, et al. B4 and After: Managing Hierarchy, Partitioning, and Asymmetry for Availability and Scale in Google's Software-Defined WAN. In *Proceedings of the 2018 Conference of the ACM Special Interest Group on Data Communication*, pages 74–87, 2018.

[24] Michael Isard, Vijayan Prabhakaran, Jon Currey, Udi Wieder, Kunal Talwar, and Andrew Goldberg. Quincy: Fair Scheduling for Distributed Computing Clusters. In *Proceedings of the ACM SIGOPS 22nd Symposium on Operating Systems Principles*, pages 261–276, 2009.

[25] George Karakostas. Faster Approximation Schemes for Fractional Multicommodity Flow Problems. *ACM Transactions on Algorithms (TALG)*, 4(1):1–17, 2008.

[26] Simon Knight, Hung X Nguyen, Nickolas Falkner, Rhys Bowden, and Matthew Roughan. The internet topology zoo. *IEEE Journal on Selected Areas in Communications*, 29(9):1765–1775, 2011.

[27] Alok Kumar, Sushant Jain, Uday Naik, Anand Raghuraman, Nikhil Kasinadhuni, Enrique Cauich Zermeno, C Stephen Gunn, Jing Ai, Björn Carlin, Mihai Amarandei-Stavila, et al. BwE: Flexible, Hierarchical Bandwidth Allocation for WAN Distributed Computing. In *Proceedings of the 2015 ACM Conference on Special Interest Group on Data Communication*, pages 1–14, 2015.

[28] Yin Tat Lee and Aaron Sidford. Efficient inverse maintenance and faster algorithms for linear programming. In *2015 IEEE 56th Annual Symposium on Foundations of Computer Science*, pages 230–249. IEEE, 2015.

[29] Xiaoqian Li and Kwan L Yeung. Traffic Engineering in Segment Routing Networks Using MILP. *IEEE Transactions on Network and Service Management*, 17(3):1941–1953, 2020.

[30] Michael Mitzenmacher. The Power of Two Choices in Randomized Load Balancing. *IEEE Transactions on Parallel and Distributed Systems*, 12(10):1094–1104, 2001.

[31] Michael Mitzenmacher and Eli Upfal. *Probability and Computing: Randomization and Probabilistic Techniques in Algorithms and Data Analysis.* Cambridge university press, 2017.

[32] Deepak Narayanan, Keshav Santhanam, Fiodar Kazhamiaka, Amar Phanishayee, and Matei Zaharia. Heterogeneity-Aware Cluster Scheduling Policies for Deep Learning Workloads. In *14th USENIX Symposium on Operating Systems Design and Implementation (OSDI 20)*, pages 481–498, 2020.

[33] Andrew Newell, Dimitrios Skarlatos, et al. RAS: Continuously Optimized Region-Wide Datacenter Resource Allocation. In *28th ACM Symposium on Operating Systems Principles*, 2021.

[34] B. O'Donoghue, E. Chu, N. Parikh, and S. Boyd. SCS: Splitting Conic Solver, version 2.1.3. https://github.com/cvxgrp/scs.

[35] B. O'Donoghue, E. Chu, N. Parikh, and S. Boyd. Conic Optimization via Operator Splitting and Homogeneous Self-Dual Embedding. *Journal of Optimization Theory and Applications*, 169(3):1042–1068, June 2016.

[36] Brendan O'Donoghue, Giorgos Stathopoulos, and Stephen Boyd. A Splitting Method for Optimal Control. *IEEE Transactions on Control Systems Technology*, 21(6):2432–2442, 2013.

[37] Adam Paszke, Sam Gross, Francisco Massa, Adam Lerer, James Bradbury, Gregory Chanan, Trevor Killeen, Zeming Lin, Natalia Gimelshein, Luca Antiga, et al. PyTorch: An Imperative Style, High-Performance Deep Learning Library. In *Advances in Neural Information Processing Systems*, pages 8024–8035, 2019.

[38] Qifan Pu, Ganesh Ananthanarayanan, Peter Bodik, Srikanth Kandula, Aditya Akella, Paramvir Bahl, and Ion Stoica. Low Latency Geo-Distributed Data Analytics. *ACM SIGCOMM Computer Communication Review*, 45(4):421–434, 2015.

[39] Ragheb Rahmaniani, Teodor Gabriel Crainic, Michel Gendreau, and Walter Rei. The Benders Decomposition Algorithm: A Literature Review. *European Journal of Operational Research*, 259(3):801–817, 2017.

[40] Sajjad Rizvi, Xi Li, Bernard Wong, Fiodar Kazhamiaka, and Benjamin Cassell. Mayflower: Improving Distributed Filesystem Performance through SDN / Filesystem Co-Design. In *2016 IEEE 36th International Conference on Distributed Computing Systems (ICDCS)*, pages 384–394. IEEE, 2016.

[41] Matthew Roughan, Albert Greenberg, Charles Kalmanek, Michael Rumsewicz, Jennifer Yates, and Yin Zhang. Experience in Measuring Backbone Traffic Variability: Models, Metrics, Measurements and Meaning. In *IMW*, 2002.

[42] Marco Serafini, Essam Mansour, Ashraf Aboulnaga, Kenneth Salem, Taha Rafiq, and Umar Farooq Minhas. Accordion: Elastic Scalability for Database Systems Supporting Distributed Transactions. *Proceedings of the VLDB Endowment*, 7(12):1035–1046, 2014.

[43] Ankit Singla, Chi-Yao Hong, Lucian Popa, and P Brighten Godfrey. Jellyfish: Networking Data Centers Randomly. In *9th USENIX Symposium on Networked Systems Design and Implementation (NSDI 12)*, pages 225–238, 2012.

[44] Lalith Suresh, João Loff, Faria Kalim, Sangeetha Abdu Jyothi, Nina Narodytska, Leonid Ryzhyk, Sahan Gamage, Brian Oki, Pranshu Jain, and Michael Gasch. Building Scalable and Flexible Cluster Managers Using Declarative Programming. In *14th USENIX Symposium on Operating Systems Design and Implementation (OSDI 20)*, pages 827–844, 2020.

[45] Rebecca Taft, Essam Mansour, Marco Serafini, Jennie Duggan, Aaron J Elmore, Ashraf Aboulnaga, Andrew Pavlo, and Michael Stonebraker. E-Store: Fine-Grained Elastic Partitioning for Distributed Transaction Processing Systems. *Proceedings of the VLDB Endowment*, 8(3):245–256, 2014.

[46] Muhammad Tirmazi, Adam Barker, Nan Deng, Md E Haque, Zhijing Gene Qin, Steven Hand, Mor Harchol-Balter, and John Wilkes. Borg: The Next Generation. In *Proceedings of the Fifteenth European Conference on Computer Systems*, pages 1–14, 2020.

[47] Alexey Tumanov, Timothy Zhu, Jun Woo Park, Michael A Kozuch, Mor Harchol-Balter, and Gregory R Ganger. TetriSched: Global Rescheduling with Adaptive Plan-Ahead in Dynamic Heterogeneous Clusters. In *Proceedings of the Eleventh European Conference on Computer Systems*, pages 1–16, 2016.

[48] Robert J Vanderbei et al. *Linear Programming*, volume 3. Springer, 2015.

[49] Wencong Xiao, Romil Bhardwaj, Ramachandran Ramjee, Muthian Sivathanu, Nipun Kwatra, Zhenhua Han, Pratyush Patel, Xuan Peng, Hanyu Zhao, Quanlu Zhang, et al. Gandiva: Introspective Cluster Scheduling for Deep Learning. In *13th USENIX Symposium on Operating Systems Design and Implementation (OSDI 18)*, pages 595–610, 2018.

[50] Matei Zaharia, Mosharaf Chowdhury, Tathagata Das, Ankur Dave, Justin Ma, Murphy McCauly, Michael J Franklin, Scott Shenker, and Ion Stoica. Resilient Distributed Datasets: A Fault-Tolerant Abstraction for In-Memory Cluster Computing. In *9th USENIX Symposium on Networked Systems Design and Implementation (NSDI 12)*, pages 15–28, 2012.

Appendix (Not Peer-Reviewed)

A Proof of Bound on Random Partitioning for Simple Allocation Problem

In this section, we show a full derivation of Equation 2, which upper bounds the probability of a large gap between the optimal solution and solution returned by POP.

To quantify the gap between POP and optimal solutions, we need a sense of how big $q_{s,t}$ – the number of misplaced jobs of type s in sub-problem t – is in practice. In this section, we assume that the number of resources of each type are not necessarily equal; we define n_s as the number of resources of type s. We can compute a probabilistic upper bound on $q_{s,t}$ using a classical Chernoff bound, interpreting the random assignment of all type-s jobs (n_s of them) to sub-problems as Bernoulli trials where the probability that any given type-r job is placed in sub-problem k is $1/k$. Define $X_{s,t}$ to be the sum of all such trials, i.e., the number of type-s jobs in sub-problem t, with $E[X_{s,t}] = n_s/k$. Note that when $X_{s,t}$ exceeds the expected value, we get $X_{s,t} = n_s/k + q_{s,t}$. The Chernoff upper bound [31] can then be used to find the upper limit on the probability that the value of $X_{s,t}$ exceeds the expected value by a fraction δ:

$$\Pr[X_{s,t} \geq (1+\delta)n_s/k] = \Pr[q_{s,t} \geq \delta n_s/k] \leq \exp\left(\frac{-\delta^2 n_s}{(2+\delta)k}\right) \quad (3)$$

In the rest of this text, to simplify notation, we will refer to the RHS of Equation 3 as $C(\delta, n_s, k)$. For a simple problem with $r = 2$ and $k = 2$, if we have $n = m = 10^5$ jobs and resources split equally across resource types, the probability of exceeding the expected amount of type A jobs in a given sub-problem by 1% is 0.2877, by 2% is 0.00694, and by 3% is 0.0000145.

This bound can be extended to misplaced jobs across all resource types and sub-problems using the union bound, i.e., $\Pr(Z_1 \vee Z_2) \leq \Pr(Z_1) + \Pr(Z_2)$. This can be used to compute an upper limit on the probability that any resource type exceeds its expectation by a fraction $(1+\delta)$ on any sub-problem. Define $Y_{s,t}$ to be the event that type-r jobs in sub-problem k are in excess of the expected amount by a factor of $(1+\delta)$, i.e., $X_{s,t} \geq (1+\delta)n_s/k$. Then, we see that the following holds:

$$\Pr[Y_{s,1} \vee ... \vee Y_{s,k}] \leq \sum_{t=1}^{k} \Pr[Y_t] \leq \sum_{t=1}^{k} C(\delta, n_s, k) \quad (4)$$

We can extend this to all resource types similarly. Let Z_r be the probability that type-s jobs in any sub-problem k exceeds $(1+\delta)n_s/k$. Using the union bound again, we can extend Equation 4 to compute the upper limit on the probability that the total number of misplaced jobs exceeds δn.

$$\Pr\left[\sum_{s=1}^{r}\sum_{t=1}^{k} q_{s,t} \geq \delta n\right] \leq \Pr[Z_1 \vee ... \vee Z_R] \leq \sum_{s=1}^{r} \Pr[Z_j] \leq \sum_{s=1}^{r}\sum_{t=1}^{k} C(\delta, n_s, k) \quad (5)$$

We can now combine this with Equation 1 to bound the performance of a randomized POP solution for the simplified allocation problem discussed in §5.1. We define Γ^* to be an optimal allocation, Γ^{POP} to be the allocation returned by the POP procedure, and $U() : \Gamma \rightarrow u$ to be a function that computes the utility of an allocation Γ. Using Equations 1 and 5, the probability that a random job partition will result in a utility that is greater than $\delta u_{\text{maxgap}} n$ from optimal is:

$$\Pr[U(\Gamma^*) - U(\Gamma^{\text{POP}}) \geq \delta u_{\text{maxgap}} n] \leq \Pr\left[\sum_{s=1}^{r}\sum_{t=1}^{k} q_{s,t} u_{\text{maxgap}} \geq \delta u_{\text{maxgap}} n\right] \leq \sum_{s=1}^{r}\sum_{t=1}^{k} C(\delta, n_s, k)$$