CS263. Homework Assignment 3
(solutions due February 15)

February 8, 2007

Exercise 1: Prove the following statement. For any boolean command b and any initial state σ such that $\sigma(x)$ is even, if $[\![\texttt{while}\ b\ \texttt{do}\ \ x := x+2]\!]\sigma = \sigma'$ and if $\sigma' \neq \bot$ then $\sigma'(x)$ is even. Unlike in previous assignment, this time you should use denotational semantics for the proof.

Exercise 2: Recall that we obtained the least fixed point of a continuous function $F : D \to D$ by starting with $\bot = F^0(\bot)$ and repeatedly applying F to the result. Let's explore what happens if we do not start with $\bot$ but with some other element $x \in D$.

1. What is a necessary and sufficient condition for the sequence $F^i(x)$ to be a chain, that is $x \sqsubseteq F(x) \sqsubseteq F^2(x) \sqsubseteq \cdots$?
2. Prove that if the condition above holds then $\bigsqcup_i F^i(x)$ is a fixed-point and furthermore it is the least of all fixed-points that are $\sqsupseteq$ than x.
3. Find a necessary and sufficient condition for the $\bigsqcup_i F^i(x)$ to be the overall least-fixed point.

Exercise 3: Prove using Hoare rules the following property: For any boolean command b, if we start the command $\texttt{while}\ b\ \texttt{do}\ \ x := x + 2$ in a state in which x is even, and if the command terminates, it terminates in a state in which x is even.

Exercise 4: Consider the following alternate Hoare rule for `while`:

$$\frac{\vdash \{A\}\ c\ \{b \Rightarrow A \wedge \neg b \Rightarrow B\}}{\vdash \{b \Rightarrow A \wedge \neg b \Rightarrow B\}\texttt{while}\ b\ \texttt{do}\ c\ \{B\}}$$

Show that the system of axioms remains complete if we replace the old rule for while with this one. You must show that any derivation that uses the old rule for while can be written with this rule instead.

Exercise 5: Consider the following alternate Hoare rule for `while`

$$\frac{\vdash \{A \wedge b\}\ c\ \{A\}}{\vdash \{A\}\texttt{while}\ b\ \texttt{do}\ c\ \{A\}}$$

This rule is not complete. Give a counterexample and a short justification.

Exercise 6: Consider now another Hoare rule for `while`:

$$\frac{\vdash \{A\}\ c\ \{A\}}{\vdash \{A\}\texttt{while}\ b\ \texttt{do}\ c\ \{A \wedge \neg b\}}$$

This rule is also incomplete. Give a counterexample and a short justification.