The transformation of linear second-order ODEs into independent second-order equations

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Introduction: modal analysis

**Coupled linear system**

\[ M \ddot{q}(t) + K q(t) = f(t) \]

**Decoupled linear system**

\[ \ddot{p}(t) + \Omega p(t) = g(t) \]

**Computation**

- Solve generalized eigenvalue problem
  \[ M u \lambda^2 = K u \]
- Diagonalize by congruence transformation
  \[ U^T M U = I \]
  \[ U^T K U = \Omega \]
Introduction: modal analysis

**Coupled linear system**

\[ M \ddot{q}(t) + Kq(t) = f(t) \]

- Standing wave solutions
- Physical profile of vibration

\[ q(t) = \sum_{j=1}^{n} u_j p_j(t) \]

**Decoupled linear system**

\[ \ddot{p}(t) + \Omega p(t) = g(t) \]
Introduction: modal analysis

**Coupled linear system**

\[ M \ddot{q}(t) + Kq(t) = f(t) \]

\[ q(t) = U \ddot{p}(t) \]

**Decoupled linear system**

\[ \ddot{p}(t) + \Omega \dot{p}(t) = g(t) \]

**Significance**

- Cornerstone in vibration analysis and structural engineering
- Physical insight leads to good approximate methods
- Model order reduction: neglect modes with little energy
- Experimental testing and system identification
Introduction: modal analysis and viscous damping

**Coupled linear system**

\[ M \ddot{q}(t) + C \dot{q}(t) + Kq(t) = f(t) \]

\[ \dot{p}(t) + D\dot{p}(t) + \Omega p(t) = g(t) \]

**Limitation**

- Modal analysis can decouple if and only if damping is classical, i.e.
  \[ CM^{-1}K = KM^{-1}C \]

- The above is necessary and sufficient for simultaneous diagonalization of three symmetric positive definite matrices
Non-classical damping

- Classical damping means that energy dissipation is uniformly distributed throughout the structure.
- Classical damping assumption is not a valid assumption for systems with two or more parts with significantly different levels of energy dissipation.
Non-classical damping in applications

Earthquake engineering

Optimal rotor control

Active vibration absorption

Compound damping matrix

Subsystem 1

Coupling

Subsystem 2
Agenda

1. Introduction

2. The decoupling problem

3. Decoupling via phase synchronization

4. Applications in structural dynamics
The decoupling problem

**Coupled linear System**

\[ M\ddot{q}(t) + C\dot{q}(t) + Kq(t) = f(t) \]

**Decoupled linear system**

\[ \ddot{p}(t) + D\dot{p}(t) + \Omega p(t) = g(t) \]

**Traditional viewpoints**

- Diagonalize by linear transformations
  - Proven to be impossible

- Triangularization by linear transformations
  - Proven to be impossible

- Approximate decoupling: errors are uncontrollable

- State space approach:
  - Structure is lost
  - Complex states without physical meaning
The decoupling problem

Coupled linear System

\[ M\ddot{q}(t) + C\dot{q}(t) + Kq(t) = f(t) \]

Decoupled linear system

\[ \ddot{p}(t) + D\dot{p}(t) + \Omega p(t) = g(t) \]

Traditional viewpoints

- Linear transformations can not work

We consider more general, perhaps nonlinear transformations
Agenda

1. Introduction
2. The decoupling problem
3. Decoupling via phase synchronization
4. Applications in structural dynamics
Review: how to solve a scalar second order ODE

Equation:

\[ m\ddot{q}(t) + c\dot{q}(t) + kq(t) = 0 \]

Ansatz:

\[ q(t) = ve^{\lambda t} \]

Algebraic equation:

\[ (m\lambda^2 + c\lambda + k)v = 0 \]

Solution of differential equation:

\[ q(t) = v_1 e^{\lambda_1 t} + v_2 e^{\lambda_2 t} \]

Complex eigenvalues: oscillations
Real eigenvalues: no oscillations
Quadratic eigenvalue problem

**Equation:**

\[ M\ddot{q}(t) + C\dot{q}(t) + Kq(t) = 0 \]

**Ansatz:**

\[ q(t) = ve^{\lambda t} \]

**Quadratic eigenvalue problem:**

\[ (M\lambda^2 + C\lambda + K)v = 0 \]

- 2n eigenvalues and corresponding eigenvectors
- Complex eigenvalues/eigenvectors in complex conjugate pairs
- Eigenvectors associated with real eigenvalues are real

**Assumption\(^*\):** if the system is non-defective, then the solution of the differential equation is:

\[ q(t) = \sum_{j=1}^{2n} v_j e^{\lambda_j t} c_j \]

* This assumption is not restrictive and can be relaxed, see D.T. Kawano, M. Morzfeld, F. Ma, JSV (2011).
The decoupled system

**Coupled linear system**

\[ M\ddot{q}(t) + C\dot{q}(t) + Kq(t) = f(t) \]

**Decoupled linear system**

\[ \ddot{p}(t) + D\dot{p}(t) + \Omega p(t) = g(t) \]

**Facts**

- Eigenvalues determine nature of system response (oscillations vs. no oscillations)
- Decoupled and original system should be isospectral
- Linear transformations are isospectral (eigenvalues are preserved)
The decoupled system

**Coupled linear system**

\[ M \ddot{q}(t) + C \dot{q}(t) + K q(t) = f(t) \]

**Fact**

Let \( M, C \) and \( K \) be square matrices and let \( M^* \) be nonsingular. A real and diagonal system, isospectral to the \( M, C, K \) system is given by \( I, D \),

\[ D = -\text{diag}(\lambda_j + \bar{\lambda}_j), \]

\[ \Omega = \text{diag}(\lambda_j \bar{\lambda}_j). \]

**Decoupled linear system**

\[ \ddot{p}(t) + D \dot{p}(t) + \Omega p(t) = g(t) \]

Isospectral systems are not necessarily connected by linear transformations

*This assumption is not restrictive and can be relaxed, see D.T. Kawano, M. Morzfeld, F. Ma, JSV (2013).*
Back to the quadratic eigenvalue problem

Equation: \[ M\ddot{q}(t) + C\dot{q}(t) + Kq(t) = 0 \]

Ansatz: \[ q(t) = ve^{\lambda t} \]

Quadratic eigenvalue problem: \[ (M\lambda^2 + C\lambda + K)v = 0 \]

Solution of differential equation: \[ q(t) = \sum_{j=1}^{2n} v_j e^{\lambda_j t} c_j \]

Damped modes:
\[ s_j(t) = v_j e^{\lambda_j t} c_j + \bar{v}_j e^{\bar{\lambda}_j t} \bar{c}_j \]
\[ = C_j e^{\alpha_j t} \begin{pmatrix} r_{j,1} \cos(\omega_j t - \phi_{j,1} - \theta_j) \\ \vdots \\ r_{j,n} \cos(\omega_j t - \phi_{j,n} - \theta_j) \end{pmatrix} \]

Free response: \[ q(t) = \sum_{j=1}^{n} s_j(t) \]
Damped modes

**Idea:**
Define a transformation on the modes to make them synchronous
Phase synchronization of damped modes

\[ CM^{-1}K \neq KM^{-1}C \]

\[ y_j(t) = \begin{pmatrix} s_{j,1}(t + \phi_{j,1}/\omega_j) \\ \vdots \\ s_{j,n}(t + \phi_{j,n}/\omega_j) \end{pmatrix} \]

Phase synchronization

\[ s_j(t) = \begin{pmatrix} y_{j,1}(t - \phi_{j,1}/\omega_j) \\ \vdots \\ y_{j,n}(t - \phi_{j,n}/\omega_j) \end{pmatrix} \]

\[ CM^{-1}K = KM^{-1}C \]
Decoupling by phase synchronization

**Coupled linear system**

\[ M \ddot{q}(t) + C \dot{q}(t) + K q(t) = 0 \]

\[ q(t) = \sum_{j=1}^{n} s_j(t) \]

\[ s_j(t) = \begin{pmatrix} \dot{y}_{j,1}(t - \phi_{j,1}/\omega_j) \\ \vdots \\ \dot{y}_{j,n}(t - \phi_{j,n}/\omega_j) \end{pmatrix} \]

\[ y_j(t) = \begin{pmatrix} s_{j,1}(t + \phi_{j,1}/\omega_j) \\ \vdots \\ s_{j,n}(t + \phi_{j,n}/\omega_j) \end{pmatrix} \]

\[ y_j(t) = z_j p_j(t) \]

\[ \ddot{p}(t) + D \dot{p}(t) + \Omega p(t) = 0 \]

**Decoupled linear system**
Decoupling by phase synchronization

**Coupled linear system**

\[ M \ddot{q}(t) + C \dot{q}(t) + K q(t) = 0 \]

**Phase synchronization**

\[ q(t) = \sum_{j=1}^{n} \text{diag}(p_j(t - \phi_{j,i}/\omega_j)) z_j \]

**Decoupled linear system**

\[ \ddot{p}(t) + D \dot{p}(t) + \Omega p(t) = 0 \]
Phase synchronization: real eigenvalues

Complex eigenvalues: pair complex conjugates

\[ s_j(t) = v_j e^{\lambda_j t} c_j + \bar{v}_j e^{\bar{\lambda}_j t} \bar{c}_j \]

Phase synchronization gives:

\[ y_j(t) = \begin{pmatrix} s_{j,1}(t + \phi_{j,1}/\omega_j) \\ \vdots \\ s_{j,n}(t + \phi_{j,n}/\omega_j) \end{pmatrix} = z_j p_j(t) \]

Pair any two distinct real eigenvalues

\[ s_j(t) = v_a e^{\lambda_a t} c_a + v_b e^{\lambda_b t} c_b \]

Phase synchronization gives: (after some algebra)

\[ y_j(t) = \begin{pmatrix} s_{j,1}(t + \phi_{j,1}/\omega_j) \\ \vdots \\ s_{j,n}(t + \phi_{j,n}/\omega_j) \end{pmatrix} = z_j p_j(t) \]
Decoupling by phase synchronization: real eigenvalues

**Coupled linear system**

\[ M \ddot{q}(t) + C \dot{q}(t) + K q(t) = 0 \]

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**Phase synchronization**

\[ q(t) = \sum_{j=1}^{n} \text{diag}(p_j(t - \phi_{j,i}/\omega_j))z_j \]

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**Decoupled linear system**

\[ \ddot{p}(t) + D \dot{p}(t) + \Omega p(t) = 0 \]
Decoupling by phase synchronization: real eigenvalues

**Complex eigenvalues**

\[(M\lambda^2 + C\lambda + K)v = 0\] has eigenvalues \(\lambda_j, \bar{\lambda}_j, j = 1, \ldots, n\)

\[(I\lambda^2 + D\lambda + \Omega)e = 0\] eigenvalues are roots of the characteristic (quadratic) equations \(\lambda_j^2 + D_{jj}\lambda_j + \Omega_{jj} = 0\). For real coefficients, complex conjugate pairs are paired up: \(D_{jj} = - (\lambda_j + \bar{\lambda}_j), \Omega_{jj} = \lambda_j \bar{\lambda}_j\)

**Real eigenvalues**

\[(M\lambda^2 + C\lambda + K)v = 0\] as eigenvalues \(\lambda_j, j = 1, \ldots, 2n\)

\[(I\lambda^2 + D\lambda + \Omega)e = 0\] Eigenvalues are roots of characteristic (quadratic) equations \(\lambda_j^2 + D_{jj}\lambda_j + \Omega_{jj} = 0\). The coefficients are real for any pairing of real eigenvalues: \(D_{jj} = - (\lambda_j + \bar{\lambda}_j), \Omega_{jj} = \lambda_j \bar{\lambda}_j\)

Nonuniqueness
Decoupling by phase synchronization: real eigenvalues

\[ \lambda_{o+1} \quad \lambda_{o+2} \quad \cdots \quad \lambda_n \quad \lambda_{n+o+1} \quad \lambda_{n+o+2} \quad \cdots \quad \lambda_{2n} \]

\[ \text{Re} \quad \text{Im} \]

\[ \lambda_o \quad \lambda_2 \quad \lambda_1 \quad \lambda_{n+1} \quad \lambda_{n+2} \quad \lambda_{n+o} \]
Decoupling by phase synchronization: inhomogeneous equation

**Coupled linear system**

\[ M \ddot{q}(t) + C \dot{q}(t) + K q(t) = f(t) \]

**Decoupled linear system**

\[ \ddot{p}(t) + D \dot{p}(t) + \Omega p(t) = g(t) \]
Decoupling by phase synchronization: inhomogeneous equation

State-space:

\[
\begin{pmatrix}
\dot{q}(t) \\
\ddot{q}(t)
\end{pmatrix} =
\begin{pmatrix}
0 & I \\
-M^{-1}K & -M^{-1}C
\end{pmatrix}
\begin{pmatrix}
q(t) \\
\dot{q}(t)
\end{pmatrix} +
\begin{pmatrix}
0 \\
M^{-1}f(t)
\end{pmatrix}
\]

Define:

\[\Lambda_1 = \text{diag}(\lambda_1, \ldots, \lambda_o, \lambda_{o+1}, \ldots, \lambda_n)\]
\[\Lambda_2 = \text{diag}(\bar{\lambda}_1, \ldots, \bar{\lambda}_n, \lambda_{n+o+1}, \ldots, \lambda_{2n})\]
\[V_1 = (v_1, \ldots, v_o, v_{o+1}, \ldots, v_n)\]
\[V_2 = (\bar{v}_1, \ldots, \bar{v}_n, v_{n+o+1}, \ldots, \bar{v}_{2n})\]

Real, invertible coordinate transformation

\[
\begin{pmatrix}
q(t) \\
\dot{q}(t)
\end{pmatrix} =
\begin{pmatrix}
V_1 & V_2 \\
V_1\Lambda_1 & V_2\Lambda_2
\end{pmatrix}
\begin{pmatrix}
I & I \\
\Lambda_1 & \Lambda_2
\end{pmatrix}^{-1}
\begin{pmatrix}
p_1(t) \\
p_2(t)
\end{pmatrix}
\]
Decoupling by phase synchronization: inhomogeneous equation

Transformed equations:

\[ p_2(t) = \dot{p}_1(t) - g_1(t) \]
\[ \ddot{p}_1(t) + Dp_1(t) + \Omega p_1(t) = (D + Id/dt)g_1(t) + g_2(t) \]

With:

\[ g_1(t) = ((V_1\Lambda_1 - V_2\Lambda_2 V_2^{-1}V_1)^{-1} + (V_2\Lambda_2 - V_1\Lambda_1 V_1^{-1}V_2)^{-1})M^{-1}f(t) \]
\[ g_2(t) = (\Lambda_1(V_1\Lambda_1 - V_2\Lambda_2 V_2^{-1}V_1)^{-1} + \Lambda_2(V_2\Lambda_2 - V_1\Lambda_1 V_1^{-1}V_2)^{-1})M^{-1}f(t) \]

Some rearranging reveals:

\[ \ddot{p}(t) + D\dot{p}(t) + \Omega p(t) = g(t) \]
\[ g(t) = (D + Id/dt)g_1(t) + g_2(t) \]
\[ q(t) = (T_1 + T_2 \ d/dt)p(t) - T_2g_1(t) \]

Nonlinear transformation

Phase synchronization

Additional time-shifts due to external force
Decoupling by phase synchronization: algorithm

Coupled system

\[ M\ddot{q} + C\dot{q} + Kq = f(t) \]

with coordinate \( q(t) \)

Solve the eigenvalue problem

\[ (M\lambda^2 + C + K)v = 0 \]

Construct

\[ \Lambda_1 = \text{diag}(\lambda_1, \ldots, \lambda_n), \quad \Lambda_2 = \text{diag}(\lambda_{n+1}, \ldots, \lambda_{2n}) \]

\[ V_1 = (v_1, \ldots, v_n), \quad V_2 = (v_{n+1}, \ldots, v_{2n}) \]

\[ T_1 = (V_1\Lambda_2 - V_2\Lambda_2)(\Lambda_2 - \Lambda_1)^{-1} \]

\[ T_2 = (V_1 - V_2)(\Lambda_2 - \Lambda_1)^{-1} \]

\[ D = -(\Lambda_1 + \Lambda_2), \quad \Omega = \Lambda_1\Lambda_2 \]

If \( f(t) = 0 \)

Set \( g(t) = 0. \)

Else

Compute \( g(t) \)

Decoupled system

\[ \ddot{p} + D\dot{p} + \Omega p = g(t) \]

with coordinate \( p(t) \)
1. Introduction
2. Problem statement
3. Decoupling via phase synchronization
4. Applications in structural dynamics
Decoupling approximation

**Coupled linear system**

\[ M\ddot{q}(t) + C\dot{q}(t) + Kq(t) = f(t) \]

\[ q(t) = Up(t) \]

**Computation**

- Solve symmetric eigenvalue problem
  \[ Mu\lambda^2 = Ku \]
- Diagonalize by congruence transformation
  \[ U^TMU = I \]
  \[ U^TCU = D \]
  \[ U^TKU = \Omega \]
- Neglect off-diagonal elements
  \[ D = \begin{pmatrix} d_{11} & d_{12} & d_{13} \\ d_{12} & d_{22} & d_{23} \\ d_{13} & d_{23} & d_{33} \end{pmatrix} \]
Response of light equipment in a base-isolated structure

**Equipment simulation**

- Linear viscoelastic model for a five-story, base-isolated building with internal equipment
- 3 DOFs, representing the displacement of the base, the primary structure and the equipment
- Base is subject to 1940 El Centro earthquake
- Decoupling by phase synchronization
- Each independent coordinate solved by RK 4.5, with coordinate dependent time-stepping
Earthquake analysis of nuclear power plant

Model order reduction

- Four interconnected rigid structures: core, pre-stressed concrete pressure vessel, basement, adjacent building
- Each structure 2 DOF (sway, rocking angle)
- Base is subject to 1940 El Centro earthquake
- Decoupling by phase synchronization
- Each independent coordinate solved by RK 4.5, with coordinate dependent time-stepping
- Low energy coordinates can be neglected to give very good 5 DOF approximation.
Model order reduction for nuclear power plant

Energy distribution among different modes generated by phase synchronization

Direct simulation

Phase synchronization: first five modes

Decoupling approximation: first five modes
Conclusions

• All viscously damped linear systems can be decoupled by phase synchronization

• The decoupling can be implemented efficiently by solving a quadratic eigenvalue problem

• Powerful model order reduction techniques can be developed

• The method can be extended to decouple systems with non-symmetric coefficient matrices
Thank you!

References: