MATH 104, final test, Dec 16th.

Name
Student ID #

All the necessary work to justify an answer and all the necessary steps of a proof must be shown clearly to obtain full credit. Partial credit **may** be given but only for significant progress towards a solution. Show all relevant work in logical sequence and indicate all answers clearly. Cross out all work you do not wish considered. Books and notes are allowed. Calculators, computers, cell phones, pagers and similar devices are not allowed during the test.

---

1. (10pts total) Construct, with proof, a compact set of real numbers whose limit points form a countable set.

2. (10pts total) Suppose that the coefficients of the power series $\sum_n a_n z^n$ are integers, infinitely many of which are distinct from zero. Prove that the radius of convergence is at most 1.

3. (10pts total, 5pts each subitem) Evaluate

$$(a) \quad \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} e^{j/n},$$

$$(b) \quad \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} (-1)^j \left(\frac{j}{n}\right)^{100}.$$

4. (10pts total) A real-valued function f defined on an interval (a, b) is said to be *convex* if

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$$

whenever $a < x, y < b$, $0 < \lambda < 1$. Prove that every convex function is continuous.

5. (10pts total, 5pts each subitem) Suppose f is a real-valued differentiable function on $\mathbb{R}$. Call x a *fixed point* of f if $f(x) = x$.

(a) If $f'(t) \neq 1$ for all $t \in \mathbb{R}$, prove that f has at most one fixed point.

(b) Show that the function $f(t) := t + (1 + e^t)^{-1}$ has no fixed points, although $0 < f'(t) < 1$ for all $t \in \mathbb{R}$.

6. (10pts total, 5pts each subitem) For x real and $n \in \mathbb{N}$, let

$$f_n(x) := \frac{x}{1 + nx^2}.$$

Show

(a) that (f_n) converges uniformly to a function f, and

(b) the equation

$$f'(x) = \lim_{n\to\infty} f_n'(x)$$

is correct for $x \neq 0$ but false for $x = 0$.