Brief solutions to the midterm.

1. The function d is a metric, since it satisfies all the metric axioms. (a) It is nonnegative for any points p and q and is zero only if $p = q$. (b) The function is symmetric. (c) It satisfies the triangle inequality. Indeed, if $p = q$, then $d(p, q) = 0$ does not exceed 0, 1 or 2, which are possible on the right-hand side. If $p \neq q$, then $d(p, q) = 1$ and the right-hand side is at least 1, since we cannot have $p = r$ and $q = r$ without having $p = q$.

The sets compact in this metric space are finite subsets F of X. They are obviously compact, since any open cover of such a set contains a finite subcover: for each element $x \in F$, just take the open set in the cover that contains x. These are the only compact sets, since if a set $S \subset X$ is infinite, then a possible open cover for S is $\cup_{x \in S}\{x\}$, which contains no finite subcover.

2. The first set is not open, closed, perfect, not connected, bounded. The second set is open, not closed, not perfect, not connected, bounded.

3. The first series diverges by comparison with the series $\sum_n 1/n$. The second converges, being an alternating series (starting when $\log n > e$). The terms of the third series can be rewritten as $1/(\sqrt[3]{(n+1)^2} + \sqrt[3]{n+1} \cdot \sqrt[3]{n} + \sqrt[3]{n^2})$ and compared to $1/n^{2/3}$. Since the series $\sum_n 1/n^{2/3}$ diverges, so does the given series.

4. Let $\varepsilon > 0$ be given. Since (p_n) is a Cauchy sequence, there exists a number $M = M(\varepsilon)$ such that

$$d(p_n, p_m) < \varepsilon/2 \quad \text{whenever } \; n,\, m > M.$$

Since the subsequence (p_{n_k}) converges to p, there exists $K = K(\varepsilon)$ such that

$$p(p_{n_k}, p) < \varepsilon/2 \qquad \text{whenever } \; k > K.$$

Let $N := \max\{n_K, M\}$. Then, for any $n > N$, we have

$$d(p_n, p) \leq d(p_n, p_{n_{K+1}}) + d(p_{n_{K+1}}, p) < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

So, $\lim_{n\to\infty} p_n = p$.