Mock final test.

1. List all limit points, supremum and infimum of the sequence $(\sin(\pi n q))$ where q is a fixed rational number. What happens for an irrational q?

2. Let (x_n) be a sequence of points in a metric space X that converges to a point x. Show that it has a subsequence converging as fast as you please. In other words, if (r_k) is any sequence of decrreasing positive numbers that tends to 0, then you can choose a subsequence (x_{n_k}) so that

$$d(x_{n_k}, x) < r_k \quad \text{for all } k \in \mathbb{N}.$$

3. Suppose f is a uniformly continuous mapping of a metric space X into a metric space Y. Prove that $(f(x_n))$ is a Cauchy sequence in Y if (x_n) is a Cauchy sequence in X.

4. Prove that the series

$$\sum_{n=1}^{\infty} (-1)^n \frac{x^2 + n}{n^2}$$

converges uniformly in every bounded interval, but does not converge absolutely for any value of x (x is meant to be real).

5. Show that deleting the terms $1/n$ for all n having digit 9 in its decimal expansion makes the harmonic series $\sum_n 1/n$ converge.

6. Suppose f is twice-differeniable on $(0, \infty)$, f'' is bounded on $(0, \infty)$, and $f(x) \to 0$ as $x \to \infty$. Prove that $f'(x) \to 0$ as $x \to \infty$.

7. Suppose α increases monotonically on $[a, b]$, g is a continuous real-valued function and $g(x) = G'(x)$ for all $x \in [a, b]$. Prove that

$$\int_a^b \alpha(x) g(x) \mathrm{dx} = G(b)\alpha(b) - G(a)\alpha(a) - \int_a^b G d\alpha.$$

8. Let (f_n) be a sequence of continuous functions which converges uniformly to a function f on a set E. Prove that

$$\lim_{n \to \infty} f_n(x_n) = f(x)$$

for every sequence of points $x_n \in E$ such that $x_n \to x \in E$. Is the converse of this true?