

## Solutions to homework # 1.

1. Let  $r$  be rational and  $x$  be irrational. If  $r + x$  were rational, then  $x = (r + x) - r$  would be rational too, as a difference of two rational numbers. Hence  $r + x$  is irrational.

Likewise, if  $rx$  were rational, then, since  $1/r$  exists and is rational, the number  $x = (rx)(1/r)$  would be rational too as a product of two rational numbers. Thus  $rx$  must be irrational.

2. Suppose there exists a rational number whose square is 12. Write such a number in lowest terms as  $m/n$ , where  $m$  and  $n$  are integers having no factors in common. Then  $m^2 = 12n^2 = 3 \cdot 2^2 \cdot n^2$ . Since 3 appears in the right-hand side, 3 divides  $m^2$ . But 3 is prime, so it must divide  $m$ ; hence  $m^2$  must be divisible by 9. But then 3 divides  $n^2$  and therefore  $n$  as well, contrary to the assumption that  $m$  and  $n$  have no common factors. Contradiction!

3. Since  $E$  is nonempty, it has at least one element, say,  $x$ . As  $\alpha$  is a lower bound of  $E$ , we get  $\alpha \leq x$ . Since  $\beta$  is an upper bound of  $E$ ,  $x \leq \beta$ . By the transitivity of the order  $\leq$  we conclude that  $\alpha \leq \beta$ .

4. Since  $A$  is nonempty and bounded below, there exists  $\inf A$ . Since  $\inf A \leq x$  for all  $x \in A$ ,  $-\inf A \geq -x$  for all  $x \in A$ , i.e.,  $-\inf A \geq y$  for all  $y \in -A$ . Thus  $-\inf A$  is an upper bound of  $-A$ . The set  $-A$  is thus nonempty and bounded above, hence there exists  $\sup(-A)$ . Since  $\sup(-A) \geq y$  for all  $y \in -A$ ,  $-\sup(-A) \leq -y$  for all  $y \in -A$ , i.e.,  $-\sup(-A) \leq x$  for all  $x \in A$ . Thus  $-\sup(-A)$  is a lower bound of  $A$ . Now,  $-\inf A \geq \sup(-A)$ ,  $-\sup(-A) \leq \inf A$  since a lower bound is not bigger than the greatest lower bound and an upper bound is not smaller than the least upper bound. But these last inequalities are equivalent to  $-\inf A \geq \sup(-A)$ ,  $-\inf A \leq \sup(-A)$ . Hence  $-\inf A = \sup(-A)$ .

5. Suppose there is an order  $>$  that makes  $\mathbb{C}$  into an ordered field. By Proposition 1.18(d), every square in  $\mathbb{C}$  must then be positive. So  $-1 = i^2 > 0$  and  $1 = 1^2 > 0$ . Adding 1 to both sides of the first inequality, we conclude (directly from the definition of an ordered field) that  $0 = 1 - 1 > 1 + 0 = 1$ , which together with  $1 > 0$  implies by transitivity that  $0 > 0$ , contrary to the first basic property of an order. So, there is no order  $<$  that makes  $\mathbb{C}$  into an ordered field.