Solutions to homework # 10.

1. If $|f(x) - f(y)| \le (x-y)^2$, then

$$0 \le \left| \frac{f(x) - f(y)}{x - y} \right| \le |x - y| \quad \text{for all } x \ne y \in \mathbb{R}.$$

Note that the leftmost and the rightmost expressions tend to 0 as $x \to y$. Therefore by the Sandwich theorem

$$\lim_{x \to y} \left| \frac{f(x) - f(y)}{x - y} \right| = 0 \quad \text{for all } y \in \mathbb{R}.$$

Hence $f' = 0$ everywhere in $\mathbb{R}$, so f is constant.

2. Define a function p as follows:

$$p(x) := \sum_{j=0}^{n} \frac{c_j x^{j+1}}{j+1} = c_0 x + \frac{c_1 x^2}{2} + \cdots + \frac{c_n x^{n+1}}{n+1}.$$

Then $p(0) = 0$ and

$$p(1) = c_0 + \frac{c_1}{2} + \cdots + \frac{c_n}{n+1} = 0$$

by the assumption of the problem. The function p is a polynomial, so in particular it is continuously differentiable everywhere in $[0, 1]$. Therefore, by Theorem 5.10, there exists $x \in (0, 1)$ such that $p'(x) = 0$, i.e.,

$$c_0 + c_1 x + \cdots + c_n x^n = 0.$$

3. Fix $x > 0$. By the assumption of the problem, f is continuous on $[0, x]$ and differentiable on $(0, x)$. Then the mean value theorem implies that there exists $c \in (0, x)$ such that $f(x) - f(0) = (x - 0) f'(c)$, hence $f(x) = x f'(c)$, i.e., $f(x)/x = f'(c)$. Since f' is monotonically increasing, the condition $c < x$ implies $f'(c) \le f'(x)$, so $f(x)/x \le f'(x)$, hence

$$0 \le \frac{x f'(x) - f(x)}{x^2} = g'(x) \quad \text{where } g(x) := \frac{f(x)}{x}.$$

Thus $g'(x) \ge 0$ for all $x > 0$, which means g is monotonically increasing.

4. Since

$$f(x) = |x|^3 = \begin{cases} x^3, & x \ge 0, \\ -x^3 & x < 0, \end{cases}$$

we get

$$f'(x) \begin{cases} 3x^2, & x > 0, \\ -3x^2 & x < 0, \end{cases} \quad \text{and } f'(0) = \lim_{x \to 0+} f'(x) = \lim_{x \to 0-} f'(x) = 0.$$

Next,

$$f''(x) = \begin{cases} 6x, & x > 0, \\ -6x & x < 0, \end{cases} \quad \text{and } f''(0) = \lim_{x \to 0+} f''(x) = \lim_{x \to 0-} f''(x) = 0.$$

But then

$$\lim_{x\to 0+}\frac{f''(x)-f''(0)}{x}=\lim_{x\to 0+}\frac{6x-0}{x}=6\neq -6=\lim_{x\to 0-}\frac{-6x-0}{x}=\lim_{x\to 0-}\frac{f''(x)-f'(0)}{x},$$

so $f'''(0)$ does not exist.

5. Apply Theorem 5.15 with $\alpha=0$, $\beta=1$, $n=3$ to get

$$f(1)=f(0)+f'(0)+f''(0)/2+f'''(s)/6 \quad \text{for some } s\in(0,1).$$

Then apply the smae theorem with $\alpha=0$, $\beta=-1$ to get

$$f(-1)=f(0)-f'(0)+f''(0)/2-f'''(t)/6 \quad \text{rm for some } t\in(-1,0).$$

Due to the assumptions of the problem, these formulas simplify to

$$1=f''(0)/2+f'''(s)/6, \quad 0=f''(0)/2-f'''(t)/6,$$

which implies $f'''(s)+f'''(t)=6$. Note that if both $f'''(s)$ and $f'''(t)$ were strictly less than 3, then their sum would be strictly less than 6, leading to a contradiction. Hence one of $f'''(s)$ and $f'''(t)$ must be greater than or equal to 3. Thus there exists $x\in(-1,1)$ such that $f'''(x)\geq 3$.