Solutions to homework # 11.

1. Since f is twice differentiable, by Theorem 5.2 f and f' are continuous on (a, ∞) and f'' exists on (a, ∞). So, for any $x \in (a, \infty)$ and $h > 0$, we apply Taylor's formula to $\alpha := x$ and $\beta = x + 2h$ and obtain

$$f(x+2h) = f(x) + 2hf'(x) + 2h^2 f''(\xi) \quad \text{for some } \xi \in (x, x+2h).$$

This yields

$$f'(x) = (f(x+2h) - f(x))/2h - hf''(\xi). \tag{1}$$

If $M_0 = \sup |f(x)|$, $M_2 = \sup |f''(x)|$, then (1) implies $|f'(x)| \leq (M_0 + M_0)/2h + hM_2$ for all $x \in (a, \infty)$. Taking the supremum over x therefore gives us $M_1 \leq M_0/h + hM_2$, with $h > 0$ being arbitrary. Taking h to be $\sqrt{M_0/M_2}$, we get

$$M_1 \leq 2\sqrt{M_0 M_2}, \quad \text{hence } M_1^2 \leq 4M_0 M_2$$

since all involved quantities are positive.

2. Pick $\varepsilon > 0$ and find $\delta > 0$ so that

$$|\alpha(x_0) - \alpha(x)| < \varepsilon/2 \quad \text{whenever } |x - x_0| \leq \delta, \ \ x \in [a, b].$$

Pick a partition $P = \{\xi_0, \dots, \xi_n\}$ of $[a, b]$ such that each subinterval has length at most δ and no ξ_i equals x_0. Then there exists a unique index k such that $x_0 \in [\xi_{k-1}, \xi_k]$, which implies $M_k = 1$, $M_i = 0$ for all $i \neq k$. So,

$$U(P, f, \alpha) = M_k \Delta\alpha_k = \alpha(\xi_k) - \alpha(\xi_{k-1}) = (\alpha(\xi_k) - \alpha(x_0)) + (\alpha(x_0) - \alpha(\xi_{k-1})) < \varepsilon$$

since α is increasing and $\xi_k > x_0 > \xi_{k-1}$. But $m_i = 0$ for all i, so $L(P, f, \alpha) = 0$. Thus $U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon$. By Theorem 6.6, the function f is therefore in $\mathcal{R}(\alpha)$ and its integral is equal to

$$\int f \, d\alpha = \inf U(P, f, \alpha) = \sup L(P, f, \alpha) = 0.$$

3. Here is a proof by contradiction. Suppose f is not identically zero on $[a, b]$. Since $f \geq 0$, then there exists a point $x_0 \in [a, b]$ such that $f(x_0) > 0$. By continuity of f, for some $\delta > 0$ we can guarantee that

$$|f(x) - f(x_0)| < \varepsilon := f(x_0)/2 \quad \text{for all } x \in U_0 := (x_0 - \delta, x_0 + \delta) \cap [a, b].$$

Then

$$f(x_0)/2 > |f(x) - f(x_0)| \geq |f(x_0)| - |f(x)|, \quad \text{hence } f(x) \geq f(x_0)/2.$$

Then, using the nonnegativity of f and additivity of integrals, we get

$$\int_a^b f(x)\mathrm{dx} = \int_{U_0} f(x)\mathrm{dx} + \int_{[a,b]\setminus U_0} f(x)\mathrm{dx} \geq \int_{U_0} f(x)\mathrm{dx} \geq \left(\frac{f(x_0)}{2}\right) \mathrm{diam}(U_0) > 0.$$

4. For every partition, its subintervals aways contain both rational and irrational points. Hence, for any partition $P = \{x_0, \ldots, x_n\}$ of $[a, b]$, we have $M_i = 1$, $m_i = 0$ for all $i = 1, \ldots, n$. Then

$$U(P, f) = \sum_{i=1}^{n} M_i \Delta x_i = \sum_{i=1}^{n} \Delta x_i = b - a \quad \text{but} \quad L(P, f) = \sum_{i=1}^{n} m_i \Delta x_i = 0.$$

So

$$\overline{\int_a^b} f \,\mathrm{dx} = \inf U(P, f) = b - a \neq 0 = \sup L(P, f) = \underline{\int_a^b} f \,\mathrm{dx},$$

therefore f is not integrable.