Solutions to homework # 12.

1. Let $f = 1$ at rational points and -1 at irrational points. Then f is real and bounded, and $f^2 = 1$ is Riemann integrable. But f itself is not integrable.

Now suppose f^3 is Riemann integrable. Consider the function $\phi(t) := t^{1/3}$. This function is defined and coninuous over all of $\mathbb{R}$. By Theorem 6.11, the function $f = \phi(f^3)$ is therefore integrable.

2. Suppose $f,\, g \in \mathcal{R}(\alpha)$. First let us prove that

$$\|f+g\|_2 \leq \|f\|_2 + \|g\|_2.$$

Using the Schwarz inequality , we have

$$\begin{aligned}
\|f+g\|_2^2 &= \int_a^b |f+g|^2\, d\alpha = \int_a^b |f|^2\, d\alpha + 2\mathrm{Re}\int_a^b f\overline{g}\, d\alpha + \int_a^b |g|^2\, d\alpha \\
&\leq \int_a^b |f|^2\, d\alpha + 2\left|\int_a^b f\overline{g}\, d\alpha\right| + \int_a^b |g|^2\, d\alpha \\
&\leq \int_a^b |f|^2\, d\alpha + 2\left(\int_a^b |f|^2\, d\alpha\right)^{1/2}\left(\int_a^b |g|^2\, d\alpha\right)^{1/2} + \int_a^b |g|^2\, d\alpha \\
&= \left((\int_a^b |f|^2\, d\alpha)^{1/2} + (\int_a^b |g|^2\, d\alpha)^{1/2}\right)^2 = (\|f\|_2 + \|g\|_2)^2.
\end{aligned}$$

So, $\|f+g\|_2 \leq \|f\|_2 + \|g\|_2$. Now replace f by $f-g$ and g by $g-h$ where h is an arbitrary function in $\mathcal{R}(\alpha)$, to get

$$\|f-h\|_2 \leq \|f-g\|_2 + \|g-h\|_2.$$

3. Let $F(x) := xf^2(x)$. Since f is continuously differentiable, so is F. By the Fundamental Theorem of Calculus,

$$\int_a^b (f^2(x) + 2xf(x)f'(x))\mathrm{dx} = F(b) - F(a) = 0$$

and, since $\int_a^b f^2(x)\mathrm{dx} = 1$, we have $\int_a^b ff(x)f'(x)\mathrm{dx} = -1/2$. By the Schwarz inequality, we therefore conclude

$$\frac{1}{4} = \left|\int_a^b xf(x)f'(x)\mathrm{dx}\right| \leq \left(\int_a^b |f'(x)|^2\right)^{1/2}\left(\int_a^b |xf'(x)|^2\mathrm{dx}\right)^{1/2}. \qquad (1)$$

The equality holds in this inequality if and only if the integrands are proportional. That means $f'(x) = cxf(x)$ for some $c \neq 0$, since f' is not identically zero. All solutions to this differential equation have the form

$$f(x) = de^{cx^2/2},$$

where d is another nonzero constant. For such a function though we cannot have $f(0) = f(1) = 0$, so equality in (1) is impossible. Thus

$$\int_a^b (f'(x))^2 \mathrm{dx} \cdot \int_a^b x^2 f^2(x) \mathrm{dx} > 1/4.$$