Solutions to homework # 13.

1. First of all, for every function f_n there exists a bound M_n such that $|f_n(x)| \leq M_n$ for all x. Now pick an arbitrary $\varepsilon > 0$. Then there exists $N = N(\varepsilon)$ such that

$$|f_n(x) - f_N(x)| < \varepsilon \quad \text{for all } x \quad \text{whenever } n > N.$$

Then $|f_n(x)| \leq \varepsilon + |f_N(x)| \leq M_N + \varepsilon$ for all $n \geq N$. Let $M := \max\{M_1, \dots, M_{N-1}, M_N + \varepsilon\}$. Then $|f_n(x)| \leq M$ for all n and all x.

2. Let f denote the limit of (f_n) and let g denote the limit if (g_n). For any $\varepsilon > 0$, there exists $N = N(\varepsilon)$ such that $|f_n(x) - f(x)| \leq \varepsilon/2$, $|g_n(x) - g(x)| \leq \varepsilon/2$ for all x. Then

$$|f_n(x) + g_n(x) - f(x) - g(x)| \leq |f_n(x) - f(x)| + |g_n(x) - g(x)| \leq \varepsilon \quad \text{forall } x.$$

If now both (f_n) and (g_n) are sequences of bounded functions, we know from Exercise 1 that they are uniformly bounded. So there exists a constant such that $|f_n(x)| \leq M$, $|g_n(x)| \leq M$ for all x and all $n \in \mathbb{N}$. Then the limits f and g also satisfy the same condition $|f(x)| \leq M$, $|g(x)| \leq M$. Given $\varepsilon > 0$, there exists $N = N(\varepsilon)$ such that $|f_n(x) - f(x)| < \varepsilon/2M$, $|g_n(x) - g(x)| < \varepsilon/2M$ for all x and all $n > N$. Then

$$\begin{aligned} |f_n(x)g_n(x) - f(x)g(x)| &\leq |(f_n(x) - f(x))g_n(x)| + |f(x)(g_n(x) - g(x))| \\ &\leq M|f_n(x) - f(x)| + M|g_n(x) - g(x)| \leq M\frac{\varepsilon}{2M} + M\frac{\varepsilon}{2M} = \varepsilon. \end{aligned}$$

3. First of all, the series is undefined at points $x = -1/k^2$ for $k \in \mathbb{N}$. So they should be excluded from our intervals. At $x = 0$ the series is just a series of ones, hence diverges. At all other points, the series converges absolutely by the Comparison test since

$$\left|\frac{1}{1 + n^2 x}\right| \leq \frac{1}{|x|} \cdot \left|\frac{1}{n^2 + 1/x}\right| \leq \frac{1}{|x|} \cdot \frac{1}{n^2 - 1/|x|}.$$

So the limit function f is defined everywhere except at points $x = 0$, $x = -1/k^2$, $k \in \mathbb{N}$.

The series converges informly on the intervals $(-\infty, -1)$, $(-1/k^2, -1/(k+1)^2)$ for each $k \in \mathbb{N}$, and on any interval of the form (a, ∞) where $a > 0$. For each of these intervals, this follows from the fact that the tail of the series is bounded by the tail of the series $1/n^2$ times a constant that depends on -1, k, or a.

Let us show that the convergence is not uniform on an interval of the form $(0, a)$ where $a > 0$. Let $\varepsilon \in (0, 1)$ be fixed and suppose N is chosen so that the tail of the series starting at N is bounded by ε. But we can find $x > 0$ so small that $1/(1 + N^2 x) > \varepsilon$, so the tail $\sum_{n \geq N} 1/(1 + n^2 x)$ is also greater than ε.

For any c where $f(c)$ is defined, it is continuous. Indeed, there exists an interval $[a, b]$ around c where the series for f converges uniformly. Since the terms of the series, hence its partial sums as well, are continuous on $[a, b]$, in particular at c.

The function f is not bounded. Assume, on the contrary, that $|f(x)| \leq M$ for all x where f is defined. Then we can choose $x > 0$ so small that $1/(1 + (2M)^2 x) > 1/2$, hence the first $2M$ terms in the series for f are each strictly larger than $1/2$, and the total is therefore strictly larger than M.