Solutions to homework # 2.

1. For arbitrary $z = a + b\mathrm{i}$, $w = c + d\mathrm{i}$, define $z < w$ if $a < c$ or $a = c$ and $b < d$. Prove that this turns $\mathbb{C}$ into an ordered set ("dictionary" or lexicographic" order). Does this ordered set have the least upper bound property?

Solution. To check that this defines an ordered set, we must establish that any two elements are comparable. Indeed, if $w \neq z$, then $a < c$, $a = c$ or $a > c$. If $a < c$, then $z < w$; if $a > c$, then $z > w$. If $a = c$, then either $b < d$ (and then $z < w$) or $b > d$ (and then $z > w$). So, any two elements of $\mathbb{C}$ are comparable using the lexicographic order.

Next, we need to check that the lexicographic order is transitive. Indeed, if $z_1 < z_2 < z_3$, then $\mathrm{Re} z_1 \leq \mathrm{Re} z_2 \leq \mathrm{Re} z_3$. If $\mathrm{Re} z_1 < \mathrm{Re} z_3$, then $z_1 < z_3$ and we are done. If $\mathrm{Re} z_1 = \mathrm{Re} z_3$, then both inequalities must be equalities, i.e., $\mathrm{Re} z_1 = \mathrm{Re} z_2 = \mathrm{Re} z_3$, but then $\mathrm{Im} z_1 < \mathrm{Im} z_2 < \mathrm{Im} z_3$, hence $z_1 < z_3$. Thus $\mathbb{C}$ is ordered by the lexicographic order.

This order does not have the least upper bound property. Consider the set $S := \{z : \mathrm{Re} z < 0\}$. This set is nonempty and bounded above by say 0. Suppose that S has a least upper bound w. Then $\mathrm{Re} w = 0$, since no number whose real part is negative is an upper bound for S, and any number whose real part is positive is greater than the upper bound 0. So, $\mathrm{Re} w = 0$. But then for any $\alpha := \mathrm{Im} w$ there are (infinitely many) complex numbers whose imaginary part is below α, which are therefore smaller than w. They are still upper bounds for S, since they have real part 0. This contradicts the defintion of the least upper bound. Thus $\mathbb{C}$ equipped with lexicographic order does not have the least upper bound property.

2. Prove that, for each $z \in \mathbb{C}$, there exists $r \geq 0$ and $w \in \mathbb{C}$ such that $|w| = 1$ and $z = rw$. Are w and r uniquely defined by z?

Solution. If $z = 0$, take $r = 0$, w any complex number of modulus 1. for $z \neq 0$, let $r := |z|$, $w := z/|z|$. The numbers r and w are uniquely determined by z except in the trivial case $z = 0$ when w is not defined uniquely. Indeed, if $r_1 w_1 = r_2 w_2 \neq 0$ and the r's and w's satisfy the properties listed above, then we get $r_1 = |r_1| \cdot |w_1| = |r_1 w_1| = |r_2 w_2| = |r_2| \cdot |w_2| = r_2 > 0$. Thus $r_1 = r_2 > 0$. Now divide the equality $r_1 w_1 = r_2 w_2$ by r_1 and obtain $w_1 = w_2$. The same argument applied to $r_1 w_1 = r_2 w_2 = 0$ shows that $r_1 = r_2 = 0$, but then w_1 and w_2 are completely free except for the conditions $|w_1| = |w_2| = 1$.

3. For any complex numbers x and y, prove that

$$||x| - |y|| \leq |x - y|.$$

Solution. Apply the triangle inequality to $x - y$ and y. We get $|(x - y) + y| \leq |x - y| + |y|$, i.e., $|x| \leq |x - y| + |y|$, i.e., $|x| - |y| \leq |x - y|$. Since x and y are completely arbitrary, they can be interchanged, which yields $|y| - |x| \leq |y - x| = |x - y|$. But then $||x| - |y|| = \max\{|x| - |y|, |y| - |x|\} \leq |x - y|$.

4. If $z \in \mathbb{C}$ and $|z| = 1$, compute $|1 + z|^2 + |1 - z|^2$.

Solution. $|1+z|^2+|1-z|^2 = (1+z)(1+\overline{z})+(1-z)(1-\overline{z}) = 1+z+\overline{z}+z\overline{z}+1-z-\overline{z}+z\overline{z} = 2+2z\overline{z} = 2+2|z|^2 = 4$.

5. Let $\vec{x}$ and $\vec{y}$ be vectors in $\mathbb{R}^k$. Show that

$$|\vec{x}+\vec{y}|^2 + |\vec{x}-\vec{y}|^2 = 2|\vec{x}|^2 + 2|\vec{y}|^2.$$

Interpret geometrically as a statement about parallelograms.

Solution. Using the dot product on the Euclidean space $\mathbb{R}^k$, we get $|\vec{x}+\vec{y}|^2 = (\vec{x}+\vec{y})\cdot(\vec{x}+\vec{y}) = \vec{x}\cdot\vec{x}+2\vec{x}\cdot\vec{y}+\vec{y}\cdot\vec{y} = |\vec{x}|^2+2\vec{x}\cdot\vec{y}+|\vec{y}|^2$. So, $|\vec{x}+\vec{y}|^2+|\vec{x}-\vec{y}|^2 = |\vec{x}|^2+2\vec{x}\cdot\vec{y}+|\vec{y}|^2+|\vec{x}|^2-2\vec{x}\cdot\vec{y}+|\vec{y}|^2 = 2|\vec{x}|^2+2|\vec{y}|^2$. If x and y are two vectors in $\mathbb{R}^k$ starting at the origin, they form a parallelogram, whose diagonals are vectors (parallel to) $\vec{x}+\vec{y}$ and $\vec{x}-\vec{y}$. This statement says that the sum of squares of diagonals in a parallelogram equals twice the sum of squares of its sides.