Solutions to homework # 3.

1. For each positive integer k, there are only finitely many equations with integer coefficients $a_0, \ldots, a_n$ such that $|a_0| + \cdots + |a_n| + n = k$. Each such equation has $n \le k$ roots. Let A_k denote all the roots of all equations with $|a_0| + \cdots + |a_n| + n = k$. Then the set of all algebraic numbers is the union $\cup_{k \in \mathbb{N}} A_k$. This is a countable union of finite sets, therefore is at most countable. The set of all algebraic numbers contains $\mathbb{Z}$, hence is infinite, so it must be countable.

2. The set $\mathbb{R}$ of all real numbers is the (disjoint) union of the sets of all rational and irrational numbers. We know from the lecture that $\mathbb{R}$ is uncountable, whereas $\mathbb{Q}$ is countable. If the set of all irrational numbers were countable, then $\mathbb{R}$ would be the union of two countable sets, hence countable. Thus the set of all irrational numbers is uncountable.

3. Here is one of many examples of a bounded set with exactly three limit points:

$$S := \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{1 + \frac{1}{n} : n \in \mathbb{N}\} \cup \{2 + \frac{1}{n} : n \in \mathbb{N}\}.$$

The set lies within the interval $[0, 3]$, therefore is bounded. The points 0, 1, 2 are the only limit points. Indeed, a limit point of S cannot be smaller than 0 or bigger than 3 because of the above bounds on S. All points in $(0, 1)$ are either isolated points of S or not in S and have a small neighborhood that does not intersect S. Likewise for all points in $(1, 2)$ and in $(2, 3]$.

4. Yes, every point of any open set E in $\mathbb{R}^2$ is a limit point of E, since every neighborhood of a point in $\mathbb{R}^2$ contains infinitely many points. Since any point in E has a neighborhood consisting of points of E only, any smaller neighborhood consists entirely of points from E as well, so every point of E is its limit point.

The same is not true for closed sets in $\mathbb{R}^2$. For example, any finite nonempty set in $\mathbb{R}^2$ is closed but none of its points is its limit point.