Solutions to homework # 4.

1. The functions d_3 and d_4 are not metrics since they do not satisfy axiom (a); for example, $d_3(1,-1) = 0$, and $d_4(2,1) = 0$. The remaining functions all satisfy (a) and (b), so it remains to see whether they satisfy the triangle inequality (c). To see that d_1 fails the triangle inequality, take $x = 1$, $y = 0$, $z = 1/2$. Thus d_1 is not a metric. The function d_2 satisfies the trianngle inequality, since the condition

$$\sqrt{|x-y|} \leq \sqrt{|x-z|} + \sqrt{|z-y|}$$

is equivalent to

$$|x-y| \leq |x-z| + |z-y| + 2\sqrt{|x-z| \cdot |z-y|}$$

and the latter is satisfied due to the triangle inequality for the absolute value function $|\cdot|$ and due to the fact that the term $2\sqrt{|x-z| \cdot |z-y|}$ is nonnegative. So, d_2 is a metric. Finally, d_5 also satisfies the triangle inequality and is therefore a metric. Indeed, the triangle inequality for d_5 is equivalent (after some algebraic manipulations) to the inequality

$$|x-y| \leq |x-z| + |z-y| + 2|x-z| \cdot |z-y| + |x-y| \cdot |x-z| \cdot |z-y|,$$

which holds because of the usual triangle inequality for $|\cdot|$.

Answer: d_1, d_3, d_4 are not metrics; d_2 and d_5 are.

2. Let $\cup_\alpha U_\alpha$ be an open cover of K. The point 0 belongs to one of the sets U_α, say U_{α_0}. Since U_{α_0} is open, it contains a neighborhood of 0 of some radius r. The points $1/n$ for $n > 1/r$ will be therefore contained in U_{α_0} as well. There only finitely many remaining points $1/n$, $n \leq 1/r$; each of them is contained in some U_{α_n}. So, the union $U_{\alpha_0} \cup_{n \leq 1/r} U_{\alpha_n}$ is a finite subcover for K. Thus, K is compact.

3. A standard example is the union $\cup_{n \in \mathbb{N},\, n>1} U_n$, where $U_n :=(1/n, 1)$. Every point $x \in (0,1)$ is in that union, since $x \in U_n$ whenever $n > 1/x$. On the other hand, if we take only finitely many intervals U_n, then their union concides with one of them, precisely, with the interval indexed by the largest integer n. That set does not contains the interval $(0, 1/n)$, so does not cover $(0,1)$.

4. The set E is uncountable, since it is in 1-1 correspondence with the set of all binary sequences (by replacing the decimal digit 4 by the binary 0 and 7 by 1). The set E is not dense in $[0,1]$. For example, the point 0.1 is not in the closure of E, since the first digit of any element of E is at least 4, so the distance of 0.1 to E is at least 0.3. Thus no sequence of elements of E can approach 0.1. The set E is perfect. Indeed, if $x \in E$ and d_n is the nth digit of the decimal expansion of x, then there is a point $y \neq x$ in E within 10^{-n+1}: just replace d_n by 7 if $d_n = 4$ or by 4 if $d_n = 7$ and leave the other digits unchanged; this defines the element y. Thus, E has no isolated points. Finally, if $x \notin E$, then its decimal expansion contains a digit other than 4 or 7. Say, it is the nth digit of x. Then the distance of x to E is positive (precisely, at least $10^{-n}(10 - (7 + 0.7 + 0.07 + \cdots))$. So, E contains all its limit points, i.e., closed. Thus E is closed and bounded, therefore compact.