Solutions to homework # 6.

1. If either lim sup on the right-hand side is $+\infty$, then the inequality is trivially satisfied. Also, if $\limsup_{n\to\infty} a_n = -\infty$, then (a_n) tends to $-\infty$; if $\limsup_{n\to\infty} b_n < \infty$, then the sequence $a_n + b_n$ tends to $-\infty$ as well. So, it remains to consider the case when both $\limsup a_n$ and $\limsup b_n$ are finite.

Take any subsequence $(a_{n_k} + b_{n_k})$ that tends to $\limsup_{n\to\infty}(a_n + b_n)$. Since (a_{n_k}) is a subsequence of (a_n), we conclude that $\limsup_{k\to\infty} a_{n_k} \leq \limsup_{n\to\infty} a_n$. Let $(a_{n_{k_l}})$ be the subsequence of a_{n_k} that tends to $\limsup_{k\to\infty} a_{n_k}$. Since $\limsup_{l\to\infty} b_{n_{k_l}} \leq \limsup_{n\to\infty} b_n$, we obtain

$$\limsup(a_n + b_n) = \lim_{l\to\infty}(a_{n_{k_l}} + b_{n_{k_l}}) \leq \limsup a_n + \limsup b_n.$$

2. (a) Since $a_n = 1/(\sqrt{n+1} + \sqrt{n}) > 1/(3\sqrt{n})$ and the series $\sum_n 1/\sqrt{n}$ diverges, we conclude by the comparison test that the series $\sum a_n$ diverges as well.

(b) Since $a_n = 1/(n(\sqrt{n+1} + \sqrt{n})) \leq 1/\sqrt{n}^3$ and the series $\sum_n 1/\sqrt{n}^3$ converges, the series $\sum_n a_n$ also converges by the comparison test.

(c) Since $\sqrt[n]{n}$ tends to 1 as n tends to ∞ and since $\sqrt[n]{n} > 1$ for all n, the numbers $\sqrt[n]{n} - 1$ are smaller than some $r \in (0,1)$ for sufficiently large n. Thus the series $\sum_n a_n$ converges by comparison with the geometric series $\sum_n r^n$.

(d) If $|z| \leq 1$, then $1/(1 + z^n) \not\to 0$ as $n \to \infty$, and so the series $\sum_n a_n$ diverges. If $|z| > 1$, then $|a_n| < 2/|z|^n$ for sufficiently large n, and the series $\sum_n a_n$ converges by comparison with the geometric series $\sum_n 1/|z|^n$.

3. We only need to establish that the sequence of partial sums is bounded, since the terms of our series are nonnegative. By the Cauchy-Schwarz inequality, we have

$$\sum_{n=1}^{N} \frac{\sqrt{a_n}}{n} \leq \sqrt{\sum_{n=1}^{N} a_n \cdot \sum_{n=1}^{N} \frac{1}{n^2}}.$$

Since the series $\sum a_n$ and $\sum 1/n^2$ are both convergent, their partial sums are bounded. Therefore, the right-hand side of the obtained inequality is bounded, and the partial sums of the series $\sum_n \sqrt{a_n}/n$ are bounded as well.

4. This setup can be reduced the the setup of Theorem 3.42. Indeed, since $\sum_n a_n$ converges, the series $\sum a_n b_n$ converges if and only if $\sum a_n(b_n - b)$ converges where b is any fixed number. By the assumption of the problem, the sequence b_n is monotone and bounded. Therefore, it has a limit; call it b. Thus $\lim_{n\to\infty}(b_n - b) = 0$. Moreover, without loss of generality we can assume that $b_n - b$ is a monotonely decreasing sequence. For if it is increasing, we can consider $(b - b_n)$ instead. Thus the sequence $b_n - b$ can be assumed to satisfy the assumptions of Theorem 3.42, hence the series $\sum_n a_n(b_n - b)$ converges, hence so does the series $\sum_n a_n b_n$.