Solutions to homework # 7.

1. (a) $\limsup_{n\to\infty} \sqrt[n]{n^3} = \lim_{n\to\infty} e^{\frac{3}{n}\log n} = e^0 = 1$, so the radius of convergence is equal to 1 according to Theorem 3.39.

(b) Theorem 3.39 can be applied here too, but this requires the fact $\lim_{n\to\infty} \frac{1}{\sqrt[n]{n!}} = 0$, which needs justification. Alternatively, one can apply the ratio test to prove that the radius of convergence is equal to ∞. Indeed, for any fixed number z,

$$\lim_{n\to\infty} \frac{(2z)^{n+1}}{(n+1)!} \cdot \frac{n!}{(2z)^n} = \lim_{n\to\infty} \frac{2z}{n+1} = 0.$$

(c) $\limsup_{n\to\infty} \sqrt[n]{2^n/n^2} = 2/\lim_{n\to\infty} n^{2/n} = 2$, since $\lim_{n\to\infty} n^{2/n} = \exp(\lim_{n\to\infty} \frac{2\log n}{n}) = e^0 = 1$, so the radius of convergence is equal to $1/2$ by Theorem 3.39.

(d) $\limsup_{n\to\infty} \sqrt[n]{n^3/3^n} = \lim_{n\to\infty} \sqrt[n]{n^3}/3 = 1/3$ by the same reasoning as in (c), hence the radius of convergence is equal to 3 by Theorem 3.39.

2. (a) First note that $x/(1+x) \geq 1/2$ whenever $x \geq 1$. So, if the sequence (a_n) has infinitely many terms greater than or equal to 1, then the sequence $(a_n/(1+a_n))$ has infinitely many terms greater than or equal to $1/2$, hence the series $\sum_n \frac{a_n}{1+a_n}$ diverges. On the other hand, if there are only finitely many such terms a_n, then, for the remaining ones, $a_n/(1+a_n) \geq a_n/2$ and the series $\sum_n \frac{a_n}{1+a_n}$ diverges as well by the comparison test.

(b) Since the partial sums s_n are nondecreasing, we see that $s_{N+j} \geq s_N$ for any N and any $j \geq 0$. Hence

$$\frac{a_{N+1}}{s_{N+1}} + \cdots + \frac{a_{N+k}}{s_{N+k}} + \frac{s_N}{s_{N+k}} \geq \frac{a_{N+1} + \cdots + a_{N+k} + s_N}{s_{N+k}} = \frac{s_{N+k}}{s_{N+k}},$$

which gives exactly

$$\frac{a_{N+1}}{s_{N+1}} + \cdots + \frac{a_{N+k}}{s_{N+k}} \geq 1 - \frac{s_N}{s_{N+k}}. \tag{1}$$

Since the partial sums s_n tend to $+\infty$, this implies that for any N, however large, there exists k such that s_N/s_{N+k} is very small, so the right-hand side of (1) is very close to 1. This means that the series $\sum_n a_n/s_n$ fails the Cauchy criterion.

(c) Since $s_n \geq s_{n-1}$ for all $n > 1$ and since $s_n - s_{n-1} = a_n$, we obtain

$$\frac{1}{s_{n-1}} - \frac{1}{s_n} = \frac{s_n - s_{n-1}}{s_{n-1}s_n} = \frac{a_n}{s_{n-1}s_n} \geq \frac{a_n}{s_n^2}.$$

Taking into account that $a_1/s_1^2 = 1/s_1$, we therefore obtain

$$\sum_{n=1}^{N} \frac{a_n}{s_n^2} \leq \frac{2}{s_1} - \frac{1}{s_N}. \tag{2}$$

Since $S_N \to \infty$ as $N \to \infty$, this shows that the right-hand side of (2) stays bounded, hence the series $\sum_n a_n/s_n^2$ converges.

(d) The first series may be either divergent or convergent. For example, if $a_n = 1/n$, then $a_n/(1+na_n) = 1/(2n)$, so the resulting series diverges. On the other hand, suppose a_n is given by the formula

$$a_n := \begin{cases} 1 & \text{if } n = 2^m,\ m \in \mathbb{Z}_+ \\ 0 & \text{otherwise.} \end{cases}$$

Then the resulting terms $a_n/(1+na_n) = 1/(1+2^m) \leq 2^{-m}$ for nonzero a_n's, so the resulting series converges. (The series $\sum_n a_n$ of course diverges. Such a series – with many zeros – is sometimes called *lacunary.*)

The second series converges by comparison with $\sum_n 1/n^2$, since, for nonzero a_n, we get $a_n/(1+n^2a_n) \leq a_n/(n^2a_n) = 1/n^2$, and $a_n/(1+n^2a_n) \leq 1/n^2$ trivially if $a_n = 0$.

3. By the triangle inequality, the terms $c_n := \sum_{k=0}^n a_k b_{n-k}$ of the Cauchy product of $\sum_n a_n$ and $\sum_n b_n$ are bounded by the terms $\tilde{c}_n := \sum_{k=0}^n |a_k| \cdot |b_{n-k}|$ of the Cauchy product of the series $\sum_n |a_n|$ and $\sum_n |b_n|$, both of which converge (absolutely). That is, $|c_n| \leq |\tilde{c}_n|$ for all n. By Theorem 3.50, the series $\sum_n \tilde{c}_n$ converges, hence the series $\sum_n |c_n|$ converges by the comparison test, i.e., the series $\sum_n c_n$ converges absolutely.