Solutions to homework # 8.

1. No, this does not imply that f is continuous. Consider the function

$$f(x) := \begin{cases} 1/|x| & \text{if } x \neq 0, \\ 0 & x = 0. \end{cases}$$

Then f is not continuous at $x = 0$ even though $f(x) - f(-x) = 0$ for all $x \in \mathbb{R}$.

2. For every $x \in E$, $f(x) \in f(E) \subseteq \overline{f(E)}$, hence $x \in f^{-1}(\overline{f(E)})$. Thus $E \subseteq f^{-1}(\overline{f(E)})$. The last set must be closed as the preimage of the closed set $\overline{f(E)}$ by the Corollary to Theorem 4.8, hence it also contains $\overline{E}$. So,

$$\overline{E} \subseteq f^{-1}(\overline{f(E)}) \Longrightarrow f(\overline{E}) \subseteq f(f^{-1}(\overline{f(E)})) \subseteq \overline{f(E)}.$$

Here is an example showing that the inclusion $f(\overline{E}) \subseteq \overline{f(E)}$ may be proper. Take X to be $\mathbb{Q}$ and Y to be $\mathbb{R}$ and let f be simply the identity map, identifying rational points as members of $\mathbb{R}$. Take E to be $\mathbb{Q}$, too. Since E is all of $\mathbb{Q}$, it is closed (in $\mathbb{Q}$), but its closure in $\mathbb{R}$ is all of $\mathbb{R}$, hence $\overline{f(E)} = \mathbb{R} \neq \mathbb{Q} = f(\overline{E})$.

3. The set $y = 0$ is closed in $\mathbb{R}$, hence its preimage $f^{-1}(0) = Z(f)$ is closed in X by the Corollary to Theorem 4.8.

4. Apply the result of Problem 2. Since $\overline{E} = X$, we get $f(X) \subseteq \overline{f(E)}$. So, the set $f(E)$ is dense in $f(X)$. Now, if two functions f and g are continuous, then so is the function $h : x \mapsto d(f(x), g(x))$. Then $h(x) = 0$ for all $x \in E$. By the result of Problem 3, the zero set $Z(h)$ of h is closed, hence $X = \overline{E} \subseteq Z(h)$. Hence $0 = h(x) = d(f(x), g(x))$ for all $x \in X$, thus $g(x) = f(x)$ for all $x \in X$.