Solutions to homework # 9.

1. By the uniform continuity of f, there exists δ such that $|f(p) - f(q)| < 1$ whenever $|p - q| < \delta$ and p, $q \in E$. Since E is bounded, its closure $\overline{E}$ is both bounded and closed, hence compact, so the following open cover of $\overline{E}$

$$\cup_{q\in\overline{E}} U_q, \quad \text{where} \quad U_q := \{p : |p - q| < \delta\}$$

has a finite subcover $\cup_{k=1}^{n} U_{p_k}$, which can also serve to cover E. Taking one point $x_k \in E$ from each U_{p_k} that contains points of E, we see that $|f(x)| \le \max_k |f(x_k)| + 1$ for all $x \in E$. Thus f is bounded on E.

The boundedness of E is indeed essential, as the example of the unformly continuous function $f(x) := x$ on the domain $E := \mathbb{R}$ demonstrates.

2. If f is continuous, then the function $g : x \mapsto f(x) - x$ is also continuous. If $f(0) \ne 0$, then $f(0) > 0$ and so $g(0) > 0$. If $f(1) \ne 1$, then $f(1) < 1$ hence $g(1) < 0$. Thus g takes values of opposite signs at the end of the interval I, hence g takes the value 0 somewhere inside I, whihc means $f(x) = x$ for some x.

3. Both functions $[x]$ and (x) are discontinuous at all integer points. Indeed, if $k \in \mathbb{Z}$ and $x \to k-$ (i.e., x tends to k from the left), then $[x] = k-1$ and $(x) \to k-(k-1) = 1$, whereas if $x \to k+$, then $[x] = k$, hence $(x) \to k - k = 0$. The functions values at the point k itself are $[k] = k$, $(k) = 0$. Thus both functions have simple discontinuities at all integer points. These are the only discontinuities, since $[x]$ stays constant on every open interval $(k-1, k)$, and the function x is continuous everywhere.

4. First let us prove that, for any irrational point x and any $n \in \mathbb{N}$, there exists a neighborhood of x such that $f(y) < 1/n$ for all points y in that neighborhood. Indeed, let d be the distance of x to the set

$$\{j/k \ : \ j \in \mathbb{Z}, \ k = 1, \ldots, n\}.$$

Since, for each $k \le n$, the points $\{j/k\}$ form a uniform mesh in $\mathbb{R}$ and since x is irrational, its distance to such a mesh is nonzero, and therefore d is nonzero, being the minimum among those distances. Then the neighborhood $N_d := \{y \ : \ |y - x| < d\}$ avoids any rational points with denominator less than or equal to n, hence $f(y) < 1/n$ on that neighborhood. Therefore, given $\varepsilon > 0$, such a neighborhood N_d can be chosen so that $|f(y)| < \varepsilon$ for all $y \in N_d$. Since $f(x) = 0$, this shows that f is continuous at (every irrational point x).

By the same argument, if $x = m/n$ is rational, its neighborhood N_d can be found so that $|f(y)| < \varepsilon$ for all $y \in N_d$, $y \ne x$. In other words, for all points y in the *punctured* neighborhood $N_d \setminus \{x\}$, we have $|f(y)| < \varepsilon$. So, both the left and the right limits $\lim_{y\to x+} f(y)$ and $\lim_{y\to x-} f(y)$ are equal to 0, but $f(x) = 1/n$, hence f has a simple discontinuity at x.