Solutions to Homework 1.

Math 110, Fall 2006.

Prob 1.2.1. (a) True – this is axiom (VS 3). (b) False – we proved the zero vector is unique. (c) False: take x to be the zero vector, and take any scalars a and b. (d) False: take a to be the zero scalar and any vectors x and y. (e) True, a column vector is also a matrix with 1 column. (f) False: it has m rows and n columns. (g) False, polynomials of different degrees can be added as in p.10. (h) False, the leading terms may cancel each other. (i) True, the leading term is multiplied by a nonzero scalar c. (j) True: the scalar c is identified with the constant function $f(x) = c$ for all x. (k) True – that's how functions are defined.

Prob 1.2.8. By two laws (VS 1) and (VS 8) of distributivity, we have

$$(a+b)(x+y) \overset{\text{VS 7}}{=\!=} a(x+y) + b(x+y) \overset{\text{VS 8}}{=\!=} ax + ay + bx + by.$$

Prob 1.2.10. First observe that the sum of two differentiable functions is differentiable, and the product of a real number and a differentiable function is also differentiable. Hence V is closed under addition and multiplication by scalars. Axioms (VS 1), (VS 2), (VS 5), (VS 6), (VS 7), (VS 8) are satisfied by functions, since they are satisfied by real numbers, and addition and scalar multiplication of functions is performed pointwise. The identically zero function 0 is differentiable, so (VS 3) holds. For a differentiable function f, its additive inverse $-f$ is differentiable as well, hence (VS 4) holds. Thus V is a vector space.

Prob 1.2.22. The field $\mathbb{Z}_2$ has 2 elements. A matrix in $M_{m\times n}(\mathbb{Z}_2)$ has m rows and n columns, so we have mn positions to fill and each entry can be chosen to take one of two possible values. So the number of all possible matrices is 2^{mn}.

Prob 1.3.1. (a) False: W may be a linear space with respect to differently defined addition and multiplication and/or over a different field. (b) False: every subspace should contain the zero vector. (c) True: just take W to be $\{0\}$. (d) False: say, intersect a circle and a line in $\mathbb{R}^2$, you will get 0, 1 or 2 points in $\mathbb{R}^2$, none of which gives you a subspace. (e) True: the only entries that may be nonzero are on the diagonal, and there are n positions there. (f) False – it is the sum of its diagonal entries.(g) False: $\mathbb{R}^2$ consists of vectors with 2 coordinates only (**Remark:** however, W it can be identified with $\mathbb{R}^2$.)

Prob 1.3.3. Let $A = (a_{ij})$, $B = (b_{ij}$. The (i,j) element of $(aA+bB)^t$ is the (j,i)th element of $aA+bB$, that is $aa_{ji} + bb_{ji}$. This is the same as the (i,j)th element of A^t multiplied by a plus the (i,j)th element of B^t multiplied by b.

Prob 1.3.9. Each is an intersection of subspaces, therefore a subspace. Solving the linear systems that define W_1, W_3 and W_4, we get

$$\begin{aligned} W_1 \cap W_3 &= \{(0,0,0)\}, \\ W_1 \cap W_4 &= W1, \\ W_3 \cap W_4 &= \{(a_1,a_2,a_3) \in \mathbb{R}^3 : 3a_1 - 11a_2 = 0,\ 11a_3 + a_1 = 0\}. \end{aligned}$$

Prob 1.3.13. Let us call the set in question W. The zero function is in W since it does take value 0 at s_0. Likewise, for any two functions f and g with $f(s_0) = g(s_0) = 0$, we also have $(f+g)(s_0) = f(s_0) + g(s_0) = 0$ and, for any scalar c, we have $(cf)(s_0) = cf(s_0) = 0$. By Theorem 1.3, we conclude that W is a subspace of $\mathcal{F}(S,F)$.

Prob 1.3.30. If $V = W_1 \oplus W_2$, then the span of W_1 amd W_2 is all of V, hence every vector $v \in V$ can be written as $v = w_1 + w_2$, $w_1 \in W_1$, $w_2 \in W_2$. If, for some vector v, there exists a different representation $\widetilde{w}_1 + \widetilde{w}_2 = v$ with $\widetilde{w}_1 \in W_1$, $\widetilde{w}_2 \in W_2$, then we get

$$w_1 - \widetilde{w}_1 = \widetilde{w}_2 - w_2.$$

Since the representations were assumed to be different, say, $w_1 \neq \widetilde{w}_1$, hence also $w_2 \neq \widetilde{w}_2$. Hence the vector $z := w_1 - \widetilde{w}_1 = \widetilde{w}_2 - w_2$ is nonzero and is simultaneously in W_1 and W_2, i.e., the intersection of W_1 and W_2 is nonempty, contrary to the sum $W_1 \oplus W_2$ being direct. Contradiction!

If $V \neq W_1 \oplus W_2$, then either W_1 and W_2 do not span V, in which case there is a vector $v \in V$ that is not expressible as a sum $w_1 + w_2$, $w_1 \in W_1$, $w_2 \in W_2$, or the intersection is nonempty. In that case, take $x \in W_1 \cap W_2$, $x \neq 0$ and consider any vector $y \in W_1$. We can produce infinitely many decompositions of y, for example $(y - x) + x = y + 0$. Notice that the first components $y + x$, y are in W_1 whereas the second components are in W_2 but these are different representations.