# Solutions to Homework 10.

### Math 110, Fall 2006.

**Prob 5.1.3.** (a) The eigenvalues are $-1$ and 4, with the eigenspaces $E_{-1} = \operatorname{span}\{[1\ -1]^t\}$, $E_4 = \operatorname{span}\{[2\ 3]^t\}$. The vectors $[1\ -1]^t$, $[2\ 3]^t$ form a basis for $\mathbb{R}^2$. The matrix $Q$ diagonalizes $A$, where

$$Q = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}, \qquad Q^{-1}AQ = \begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix}.$$

(b) The eigenvalues are 3, 2 and 1, with the eigenspaces $E_1 = \operatorname{span}\{[1\ 1\ -1]^t\}$, $E_2 = \operatorname{span}\{[1\ -1\ 0]^t\}$, $E_3 = \operatorname{span}\{[1\ 0\ -1]^t\}$. These three vectors vectors form a basis for $\mathbb{R}^3$. The matrix $Q$ diagonalizes $A$, where

$$Q = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ -1 & 0 & -1 \end{bmatrix}, \qquad Q^{-1}AQ = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

(c) The eigenvalues are 1 and $-1$, with the eigenspaces $E_1 = \operatorname{span}\{[1+i\ \ 2]^t\}$, $E_{-1} = \operatorname{span}\{[-1+i\ \ 2]^t\}$. These two vectors form a basis for $\mathbb{C}^2$. The matrix $Q$ diagonalizes $A$, where

$$Q = \begin{bmatrix} 1+i & -1+i \\ 2 & 2 \end{bmatrix}, \qquad Q^{-1}AQ = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

(d) The eigenvalues are 1 (with multiplicity two) and 0, with the eigenspaces $E_1 = \operatorname{span}\{[0\ 1\ 0]^t, [1\ 0\ 1]^t\}$, $E_0 = \operatorname{span}\{[1\ 4\ 2]^t\}$. These three vectors vectors form a basis for $\mathbb{R}^3$. The matrix $Q$ diagonalizes $A$, where

$$Q = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 4 \\ 0 & 1 & 2 \end{bmatrix}, \qquad Q^{-1}AQ = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

**Prob 5.1.22.** (a) and (b) If $Tx = \lambda x$, then $T^j x = \lambda^j x$, hence, for any polynomial $g(t) = a_0 + a_1 t + \cdots + a_m t^m$, we get

$$(a_0 I + a_1 T + \cdots + a_m T^m)x = a_0 x + a_1 \lambda x + \cdots + a_m \lambda^m x = (a_0 + a_1 \lambda + \cdots + a_m \lambda^m)x = g(\lambda)x.$$

(c) $$\left(2\begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}^2 - \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 14 & 10 \\ 15 & 19 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = 29\begin{bmatrix} 2 \\ 3 \end{bmatrix} = (2 \cdot 4^2 - 4 + 1)\begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

**Prob 5.2.1.** (a) False: take the identity map. (b) False: they can be multiples of each other. (c) False: the zero vector is not an eigenvector. (d) True (by Theorem 5.5). (e) True (by the Corollary to Theorem 5.5). (f) False: the characteristic polynomial must also split into linear factors. (g) True due to the complete splitting into linear factors. (h) True (follows from the defintion of a direct sum). (i) False: checking pairwise intersections is not enough, one needs to check intersections of each $W_j$ with the sum of the remaining $W_k$'s. For example, take three distinct lines through the origin in $\mathbb{R}^2$ as $W_1$, $W_2$ and $W_3$, then the intersection of any two of them is $\{0\}$, but their sum is not direct.

**Prob 5.2.8.** Since $\lambda_2$ is an eigenvalue, its geometric multiplicity ($\dim E_{\lambda_2}$) cannot be below 1. On the other hand, $\dim E_{\lambda_1} + \dim E_{\lambda_2} \le n$, and $\dim E_{\lambda_1} = n - 1$, so $\dim E_{\lambda_2} = 1$. Since algebraic multiplicity is at least as large as the geometric multiplicity and the algebraic multiplicities of $\lambda_1$ and $\lambda_2$ sum up to at most $n$ as well, we conclude that the geometric multiplicity of each $\lambda_j$ coincides with its geometric multiplicity $\dim E_{\lambda_j}$, for $j = 1, 2$. Since the degree of the characteristic polynomial is $n$, we conclude that it splits completely as

$$\det(\lambda I - A) = (\lambda - \lambda_1)^{n-1}(\lambda - \lambda_2).$$

So, $A$ meets the diagonalizability criterion.

**Prob 5.2.11.** Since $A$ is similar to an upper-triangular matrix with eigenvalues $\lambda_1, \ldots, \lambda_k$ repeated $m_1, \ldots, m_k$ times, respectively, the trace and the determinant of $A$ are the same as the trace and the determinant of that upper-triangular matrix. Hence

$$\operatorname{tr}(A) = m_1\lambda_1 + m_2\lambda_2 + \cdots + m_k\lambda_k, \qquad \text{and} \qquad \det(A) = \lambda_1^{m_1}\lambda_2^{m_2}\cdots\lambda_k^{m_k}.$$

**Prob 5.3.1.** (a) True: follows from the fact $(QAQ^{-1})^m = QA^mA^{-1}$. (b) True (by Theorem 5.13). (c) False, the entries must also be nonnegative. (d) False, it is the column sums that are equal to 1. (e) True (Corollary to Theorem 5.15). (f) True: $\det(3I - A) = 8 - (2z + 3z^2 - z^3)$, and since $|z| < 1$, we have $|\det(3I - A)| \geq 8 - |2z + 3z^2 - z^3| \geq 8 - |2z| - |3z^2| - |z^3| \geq 8 - 2 - 3 - 1 = 2$, so 3 cannot be an eigenvalue of $A$. (g) True (Theorem 5.17). (h) False: for example, take

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

(i) False: e.g., take the matrix $A$ from (h), then even powers of $A$ are equal to to $I$ but its odd powers are equal to $A$, so the limit $\lim_{n\to\infty} A^n$ does not exist. (j) True (by Theorem 5.20).

**Prob 5.3.2.** (a) The eigenvalues of $A$ are $-0.6$ and $0.8$, so $\lim_{n\to\infty} A^n = 0$. (b) The eigenvalues of $A$ are $-0.6$ and 1, and $A$ is diagonalizable as follows:

$$Q^{-1}AQ = \begin{bmatrix} -0.6 & 0 \\ 0 & 1 \end{bmatrix}, \qquad \text{where} \quad Q = \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix}.$$

This implies that

$$\lim_{n\to\infty} A^n = Q \lim_{n\to\infty} \begin{bmatrix} -0.6 & 0 \\ 0 & 1 \end{bmatrix}^n Q^{-1} = Q \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} Q^{-1} = \begin{bmatrix} -0.5 & 0.5 \\ -1.5 & 1.5 \end{bmatrix}.$$

(c) The eigenvalues of $A$ are $-0.3$ and 1 and $A$ is diagonalizable via

$$Q^{-1}AQ = \begin{bmatrix} -0.3 & 0 \\ 0 & 1 \end{bmatrix}, \qquad \text{where} \quad Q = \begin{bmatrix} 1 & 7 \\ -1 & 6 \end{bmatrix}.$$

So, as above,

$$\lim_{n\to\infty} A^n = Q \lim_{n\to\infty} \begin{bmatrix} -0.3 & 0 \\ 0 & 1 \end{bmatrix}^n Q^{-1} = Q \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} Q^{-1} = \begin{bmatrix} 7/13 & 7/13 \\ 6/13 & 6/13 \end{bmatrix}.$$

(d) The eigenvalues of $A$ are $-0.2$ and $0.6$, so $\lim_{n\to\infty} A^n = 0$. (e) The eigenvalues of $A$ are $-1$ and 2, so $\lim_{n\to\infty} A^n$ does not exist. (f) The eigenvalues of $A$ are 1 and 0.5, and $A$ is diagonalizable via

$$Q^{-1}AQ = \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix}, \qquad \text{where} \quad Q = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}.$$

This implies

$$\lim_{n\to\infty} A^n = Q \lim_{n\to\infty} \begin{bmatrix} -0.3 & 0 \\ 0 & 1 \end{bmatrix}^n Q^{-1} = Q \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} Q^{-1} = \begin{bmatrix} 3 & -1 \\ 6 & -2 \end{bmatrix}.$$

(g) The eigenvalues of $A$ are 1, 1, and $-0.4$, and $A$ is diagonalizable via

$$Q^{-1}AQ = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -0.4 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \text{where} \quad Q = \begin{bmatrix} 0 & 1 & 4 \\ 1 & 2 & 1 \\ 0 & -1 & -8 \end{bmatrix}.$$

This implies

$$\lim_{n\to\infty} A^n = Q \lim_{n\to\infty} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -0.4 & 0 \\ 0 & 0 & 1 \end{bmatrix}^n Q^{-1} = Q \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} Q^{-1} = \begin{bmatrix} -1 & 0 & -1 \\ -4 & 1 & -2 \\ 2 & 0 & 2 \end{bmatrix}.$$

(h) The eigenvalues of $A$ are $-0.8$, $0.5$ and $1$. The matrix $A$ is diagonalizable via

$$Q^{-1}AQ = \begin{bmatrix} -0.8 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \text{where} \quad Q = \begin{bmatrix} 0 & 1 & 4 \\ 1 & 2 & 1 \\ 0 & -1 & -8 \end{bmatrix}.$$

This implies

$$\lim_{n\to\infty} A^n = Q \lim_{n\to\infty} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -0.4 & 0 \\ 0 & 0 & 1 \end{bmatrix}^n Q^{-1} = Q \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} Q^{-1} = \begin{bmatrix} -2 & -3 & -1 \\ 0 & 0 & 0 \\ 6 & 9 & 3 \end{bmatrix}.$$

(i) The eigenvalues of $A$ are $\pm i$ and $1/2$, so $\lim_n A^n$ does not exist. (j) The eigenvalues of $A$ are $-14/3 - 7/3\,i$, $2/3 - 1/3\,i$, $-1/2$, so $\lim_{n\to\infty} A^n$ does not exist.

**Prob 5.3.3.** If $\lim_{m\to\infty} A_m = L$, then, for each position $(i,j)$, the limit $\lim_{m\to\infty} A_m(i,j) = L(i,j)$. Hence $\lim_{m\to\infty} A^t_m(i,j) = \lim_{m\to\infty} A_m(j,i) = L(j,i) = L^t(i,j)$ for all $i$, $j$. Hence $\lim_{m\to\infty} A^t_m = L^t$.

**Prob 5.3.4.** If $A$ is diagonalizable, then $A = QDQ^{-1}$ where

$$D = \begin{bmatrix} d_1 & 0 & 0 & \cdots & 0 \\ 0 & d_2 & 0 & \cdots & 0 \\ 0 & 0 & d_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & d_n \end{bmatrix}$$

is a diagonal matrix. Since $\lim_{n\to\infty} A^n$ exists, each diagonal entry $d_j$ of $D$ is either equal to 1 or is strictly below 1 in absolute value. If $|d_j| < 1$ for some $j$, then $\lim_{n\to\infty} D^n(j,j) = 0$, hence $\operatorname{rank}(\lim_{n\to\infty} D^n) < n$, so

$$\operatorname{rank}(\lim_{n\to\infty} A^n) = \operatorname{rank}(Q \lim_{n\to\infty} D^n Q^{-1}) < n.$$

Otherwise all $d_j = 1$, hence $D = I$, hence $A = I$, and therefore $\lim_{n\to\infty} A^n = I$.

**Prob 5.3.13.** The transition matrix for this Markov chain is

$$M = \begin{bmatrix} 0.7 & 0.1 & 0 \\ 0.3 & 0.7 & 0.1 \\ 0 & 0.2 & 0.9 \end{bmatrix},$$

and the initial probability vector is $p = [0.4\ 0.2\ 0.4]^t$. The percentages of Americans who own cars of each size in 1995 is given by the vector $M^2p = [0.24\ 0.34\ 0.42]^t$, and the eventual percentages by $\lim_{n\to\infty} M^n p$. Since

$$Q^{-1}MQ = \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 0.8 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \text{where} \quad Q = \begin{bmatrix} 1 & 1 & 1 \\ -2 & 1 & 3 \\ 1 & -2 & 6 \end{bmatrix},$$

we get

$$\lim_{n\to\infty} M^n = Q \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} Q^{-1} = \begin{bmatrix} 1/10 & 1/10 & 1/10 \\ 3/10 & 3/10 & 3/10 \\ 3/5 & 3/5 & 3/5 \end{bmatrix},$$

hence the eventual percentages are $\lim_{n\to\infty} M^n p = [0.1\ 0.3\ 0.6]^t$.