Solutions to Homework 12.

Math 110, Fall 2006.

Prob 6.1.1. **(a)** True, directly by definition. **(b)** True, this is also required by the definition. **(c)** False, it is linear in the first component and conjugate linear in the second component. **(d)** False, there are in fact infinitely many inner products. **(e)** False, it also holds in infinite- dimensional spaces (see the proof, which does not use finite-dimensionality). **(f)** False, every rectangular matrix has a conjugate transpose (defined the usual way). **(g)** False when x is fixed. **(h)** True when x runs over the whole space.

Prob 6.1.3. We have:

$$\begin{aligned}
\langle f,g\rangle &= \int_0^1 te^t dt = (t-1)e^t\Big|_0^1 = 1,\\
\|f\|^2 &= \int_0^1 t^2 dt = \frac{t^3}{3}\Big|_0^1 = \frac{1}{3}, \quad \text{so } \|f\| = \frac{1}{\sqrt{3}},\\
\|g\|^2 &= \int_0^1 e^{2t} dt = \frac{e^{2t}}{2}\Big|_0^1 = \frac{e^2-1}{2}, \quad \text{so } \|g\| = \sqrt{\frac{e^2-1}{2}},\\
\|f+g\|^2 &= \int_0^1 (t+e^t)^2 dt = \int_0^1 (t^2+2e^t+e^{2t})dt = \frac{3e^2+11}{6}, \quad \text{so } \|f+g\| = \sqrt{\frac{3e^2+11}{6}}.
\end{aligned}$$

Since $e^2 > 7$, we can confirm that the Cauchy-Schwarz inequality holds: $|\langle f,g\rangle| = 1 \le \sqrt{\frac{e^2-1}{6}}$. Likewise,

$$\|f+g\| = \sqrt{\frac{3e^2+11}{6}} \le \frac{1}{\sqrt{3}} + \sqrt{\frac{e^2-1}{2}} = \|f\| + \|g\|.$$

The intermediate inequality can be checked numerically or by squaring both sides:

$$\frac{1}{3} + \frac{e^2-1}{2} + 2\sqrt{\frac{e^2-1}{6}} = \frac{3e^2-1}{6} + 2\sqrt{\frac{e^2-1}{6}} \ge \frac{3e^2-1}{6} + 2 = \frac{3e^2+11}{6}.$$

Prob 6.1.9. **(a)** Take $z = x$; we get $\langle x,x\rangle = 0$. By the last axiom for the inner product, this implies $x = 0$.

(b) By linearity in the first argument, we get $0 = \langle x,z\rangle - \langle y,z\rangle = \langle x-y,z\rangle$ for all z. So, the problem reduces to the one above. By the result of (a), we conclude that $x - y = 0$, i.e., $x = y$.

Prob 6.1.17. To show that T is 1-to-1, we must check that the kernel of T consists of 0 only. Indeed, if $Tx = 0$, then $0 = \|Tx\| = \|x\|$. But $\|x\| = 0$ if and only if $x = 0$, so we are done.

Prob 6.2.2ab. **(a)** The orthonormal basis obtained by using Gram-Schmidt orthogonalization consists of vectors $(1,0,1)/\sqrt{2}$, $(-1,2,1)/\sqrt{6}$, $(-1,-1,1)/\sqrt{3}$. The coordinates of the vector $(1,1,2)$ in this basis are its inner products with the basis vectors, i.e., $3/\sqrt{2}$, $3/\sqrt{6}$, 0.

(b) The orthonormal basis gotten by using Gram-Schmidt here is $(1,1,1)/\sqrt{3}$, $(-2,1,1)/\sqrt{6}$, $(0,-1,1)/\sqrt{2}$. The coordinates of the vector $(1,0,1)$ in this basis are $2/\sqrt{3}$, $-1/\sqrt{6}, 1/\sqrt{2}$.

Prob 6.2.9. A basis for W consists of one vector, which can be taken to the normalized initial vector: $(i,0,1)/\sqrt{2}$. The subspace $W^\perp$ is 2-dimensional and consists of all vectors (x_1,x_2,x_3) such that $-ix_1+x_3 = 0$. A possible orthonormal basis for $W^\perp$ is $\{(0,1,0),(1,0,i)/\sqrt{2}\}$ (but there are infinitely many other choices).

Prob 6.2.15. **(a)** Since the basis $\{v_1,\ldots,v_n\}$ is orthonormal, the vectors x and y can be written as

$$x = \sum_{i=1}^n \langle x,v_i\rangle v_i, \qquad y = \sum_{j=1}^n \langle y,v_j\rangle v_j.$$

Multiplication of x and y gives

$$\langle x,y\rangle = \sum_i\sum_j \langle x,v_i\rangle\overline{\langle y,v_j\rangle}\langle v_i,v_j\rangle = \sum_{i,j}\langle x,v_i\rangle\overline{\langle y,v_j\rangle}\delta_{ij} = \sum_i \langle x,v_i\rangle\overline{\langle y,v_i\rangle}.$$

(b) The entries $\langle x,v_i\rangle$, $i=1,\ldots,n$ are the coordinates of x in the basis β; the same holds for the entries $\langle y,v_i\rangle$ and the vector y. The right-hand side of the formula

$$\langle x,y\rangle = \sum_i \langle x,v_i\rangle\overline{\langle y,v_i\rangle}$$

is therefore the standard inner product of $[x]_\beta$ and $[y]_\beta$, i.e., the standard inner product of $\phi_\beta(x)$ and $\phi_\beta(y)$.

Prob 6.2.16. **(a)** Let $v := \sum_{i=1}^n \langle x,v_i\rangle v_i$. Then

$$0 \le \|x-v\|^2 = \langle x,x\rangle - 2\mathrm{Re}\langle x,v\rangle + \langle v,v\rangle.$$

On the other hand,

$$\begin{aligned}\langle x,v\rangle &= \langle x, \sum_i \langle x,v_i\rangle v_i\rangle = \sum_i \overline{\langle x,v_i\rangle}\langle x,v_i\rangle = \sum_i |\langle x,v_i\rangle|^2,\\ \langle v,v\rangle &= \sum_{i,j}\langle x,v_i\rangle\overline{\langle x,v_j\rangle}\langle v_i,v_j\rangle = \sum_{i,j}\langle x,v_i\rangle\overline{\langle x,v_j\rangle}\delta_{ij} = \sum_i |\langle x,v_i\rangle|^2.\end{aligned}$$

So, $0 \le \|x-v\|^2 = \|x\|^2 - \sum_{i=1}^n |\langle x,v_i\rangle|^2$, QED.

(b) The proof in (a) shows that Bessel's inequality is an equality if and only if $x = v = \sum_{i=1}^n \langle x,v_i\rangle v_i$. In turn, this decomposition is valid if and only if $x \in \operatorname{span}\{v_j : j=1,\ldots,n\}$.

Prob 6.3.3. **(a)** $T = T^*$ since the matrix of T in the standard basis is symmetric. Hence $T^*(3,5) = T(3,5) = (11,-12)$.

(b) The matrix of T^* in the standard basis is the conjugate transpose of the matrix of T, which gives $T^*(z_1,z_2) = (2z_1 + (1+i)z_2, -iz_1)$. Hence $T^*(3-i, 1+2i) = (5+i, -1-3i)$.

(c) We must find T^*f such that

$$\int_{-1}^1 (T^*f)(t)g(t)dt = \int_{-1}^1 f(t)Tg(t)dt \quad \text{for all } \; g \in \mathcal{P}_1(\mathbb{R}).$$

Let $g(t) = b_0 + b_1 t$. Note that b_0, b_1 are arbitrary real numbers. Then $(Tg)(t) = (3b_0 + b_1 + 3b_1 t)$, so we must match

$$\int_{-1}^1 (T^*f)(t)(b_0 + b_1 t)dt = \int_{-1}^1 (4-2t)(3b_0 + b_1 + 3b_1 t)dt = 24b_0 + 4b_1.$$

Since $(T^*f)(t) = a_0 + a_1 t$, we can now find the coefficients a_0 and a_1 by choosing $b_0 = 1$, $b_1 = 0$ and $b_0 = 0$, $b_1 = 1$. This gives $(T^*f)(t) = 12 + 6t$.

Prob 6.3.12. **(a)** A vector v belongs to $R(T^*)^\perp$ if and only if $\langle v, T^*y\rangle = 0$ for all $y \in V$. This is equivalent to $\langle Tv, y\rangle = 0$ for all $y \in V$. This is equivalent, by the result of Prob. 6.1.9a, that $Tv = 0$. So, $R(T^*)^\perp = \ker T$.

(b) Since $R(T^*)^\perp = \ker(T)$ by Result (a), the orthogonal complements are also equal: $(R(T^*)^\perp)^\perp = \ker(T)^\perp$. In a finite-dimensional space, any subspace W satisfies $(W^\perp)^\perp = W$, so we obtain

$$R(T^*) = (R(T^*)^\perp)^\perp = \ker(T)^\perp.$$

Remark: Here is, for the record, the proof that $(W^\perp)^\perp = W$: Since $W \oplus W^\perp = V$, $\dim W \oplus \dim W^\perp = \dim V$. By the same token, $W^\perp \oplus (W^\perp)^\perp = V$, so $\dim W + \dim (W^\perp)^\perp = V$. But now $W \subseteq (W^\perp)^\perp$ and the dimensions of W and $(W^\perp)^\perp$ coincide. Hence $W = (W^\perp)^\perp$.