Solutions to Homework 3.

Math 110, Fall 2006.

Prob 2.1.10. By the linearity of T, we use the fact $-(1,0)+3(1,1)=(2,3)$ to obtain

$$T(2,3) = -T(1,0) + 3T(1,1) = -(1,4) + 3(2,5) = (5,11).$$

The map T is $1-1$ as we see that $T(a(1,0)+b(1,1)) = a(1,4)+b(2,5)$ implies $a=b=0$. Thus $N(T)=\{0\}$, so T is $1-1$.

Prob 2.1.15. The map T is linear, since indefinite integration is linear:

$$\int_0^x (\alpha f(t) + \beta g(t))dt = \alpha \int_0^x f(t)dt + \beta \int_0^x g(t)dt.$$

To see that T is onto, suppose Tf is the identically zero polynomial. By differentiating Tf, we obtain $f=0$. Hence the kernel of T is trivial. Finally, notice that all polynomials of the form Tf are zero at $x=0$. So we cannot get any polynomial with nonzero constant term by applying T, i.e., T is not onto.

Prob 2.1.28. T maps 0 to 0 by linearity and maps V to V by definition, these two subspaces are trivially T-invariant. If $x \in N(T)$, then $Tx = 0 \in N(T)$, so $N(T)$ is T-invariant. If $x \in R(T)$, then $Tx \in R(T)$ as the image of x, hence $R(T)$ is T-invariant as well.

Prob 2.1.35.

(a) Suppose $V = R(T) + N(T)$. Then $V = \operatorname{span}\{\beta, \gamma\}$ where β is a basis for $R(T)$ and γ is a basis for $N(T)$. We know by the Rank-Nullity formula (using finite-dimensionality of V) that $\dim V = \dim R(T) + \dim N(T)$. Hence the total number of vectors in the spanning set $\alpha \cup \gamma$ is exactly equal to the dimension of V. Hence by Corollary 2 to the Replacement Theorem, the set $\alpha \cup \gamma$ is a basis for V, hence no nontrivial linear combination of vectors β is equal to a nontrivial linear combination of vectors γ, i.e., the subspaces $R(T)$ and $N(T)$ intersect only at $\{0\}$. Thus $R(T) \oplus N(T) = V$.

(b) Suppose $R(T) \cap N(T) = \{0\}$. Pick a basis β for $R(T)$ and γ for $N(T)$. Since no nontrivial linear combination of vectors β is equal to a nontrivial linear combination of vectors γ, we conclude that the set $\alpha \cup \beta$ is linearly independent. We know by the Rank-Nullity formula (using finite-dimensionality of V) that $\dim V = \dim R(T) + \dim N(T)$. Hence the total number of vectors in the linearly independent set $\alpha \cup \gamma$ is exactly equal to the dimension of V. Hence by Corollary 2 to the Replacement Theorem, the set $\alpha \cup \gamma$ is a basis for V. Thus $R(T) \oplus N(T) = V$.

Prob 2.2.3.

$$[T]_\beta^\gamma = \begin{bmatrix} -1/3 & -1 \\ 0 & 1 \\ 2/3 & 0 \end{bmatrix}, \qquad [T]_\alpha^\gamma = [T]_\beta^\gamma [Id]_\alpha^\beta = \begin{bmatrix} -1/3 & -1 \\ 0 & 1 \\ 2/3 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} -7/3 & -11/3 \\ 2 & 3 \\ 2/3 & 4/3 \end{bmatrix}.$$

Prob 2.2.10. By listing images of each basis vector, we obtain an upper-triangular matrix of order n with 1's on the main and first upper diagonals.

Prob 2.2.13. Suppose $\alpha T + \beta U = 0$. Then $\alpha Tx + \beta Ux = 0$, i.e., $T(\alpha x) = U(\beta x)$ for all $x \in V$. Since the intersection of $R(T)$ and $R(U)$ is trivial, we conclude that $T(\alpha x) = U(\beta x) = 0$ for all $x \in V$. If α are β are nonzero scalars, then αx or βx runs over all of V, hence the corresponding map (T or U) maps every vector from V to zero. That would mean the map is the zero map, contrary to our assumption. Hence both α and β must be zero.

Prob 2.3.2.

$$(\mathbf{a}) \qquad A(2B+3C) = \begin{bmatrix} 20 & -9 & 18 \\ 5 & 10 & 8 \end{bmatrix}, \quad (AB)D = A(BD) = \begin{bmatrix} 29 \\ -26 \end{bmatrix}.$$

$$(\mathbf{b}) \qquad A^t = \begin{bmatrix} 2 & -3 & 4 \\ 5 & 1 & 2 \end{bmatrix}, \quad A^tB = A(BD) = \begin{bmatrix} 23 & 19 & 0 \\ 26 & -1 & 10 \end{bmatrix}, \quad BC^t = \begin{bmatrix} 12 \\ 16 \\ 29 \end{bmatrix},$$

$$CB = \begin{bmatrix} 27 & 7 & 9 \end{bmatrix}, \quad CA = \begin{bmatrix} 20 & 26 \end{bmatrix}.$$

Prob 2.3.3.

$$(\mathbf{a}) \qquad [U]_\beta^\gamma = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}, \quad [T]_\beta = \begin{bmatrix} 2 & 3 & 0 \\ 0 & 3 & 6 \\ 0 & 0 & 4 \end{bmatrix}, \quad [UT]_\beta^\gamma = \begin{bmatrix} 2 & 6 & 6 \\ 0 & 0 & 4 \\ 2 & 0 & -6 \end{bmatrix}.$$

$$(\mathbf{b}) \qquad [h]_\beta = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}, \quad [U(h(x))]_\gamma = \begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix}.$$