Solutions to Homework 4.

Math 110, Fall 2006.

Prob 2.3.13. Let $A = [a_{ij}]$. Then $A^t = [a_{ji}]$, and

$$\operatorname{tr}(A) = \sum_{i=1}^{n} a_{ii} = \sum_{j=1}^{n} a_{jj} = \operatorname{tr}(A^t).$$

The elements of $A = [a_{ij}]$, $B = [b_{ij}]$, $AB = [c_{ij}]$ and $BA = [d_{ij}]$ are connected by the formulas

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}, \qquad d_{ij} = \sum_{k=1}^{n} b_{ik} a_{kj}.$$

So,

$$\operatorname{tr}(AB) = \sum_{i=1}^{n} c_{ii} = \sum_{i=1}^{n} \sum_{k=1}^{n} a_{ik} b_{ki} = \sum_{i=1}^{n} \sum_{k=1}^{n} b_{ik} a_{ki} = \sum_{i=1}^{n} d_{ii} = \operatorname{tr}(BA).$$

The second-last equality is obtained by interchanging indices i and k.

Prob 2.3.15. Suppose A has n columns $\{A_i : i = 1, \ldots, n\}$, and write $A = [A_1, A_2, \ldots, A_n]$. Then $MA = [MA_1, \ldots, MA_n]$, i.e., the columns of MA are column-vectors $MA_1, \ldots, MA_n$. So if $A_j = \sum_{i \neq j} a_i A_i$, then

$$MA_j = M(\sum_{i \neq j} a_i A_i) = \sum_{i \neq j} a_i MA_i,$$

i.e., the jth column of MA is a linear combination of its other columns with the same coefficients.

Prob 2.3.17. For any vector x, we have $x = Tx + (x - Tx)$. Note that Tx is in the range $R(T)$ of T and $x - Tx$ is in the kernel $N(T)$, since

$$T(x - Tx) = Tx - T^2 x = Tx - Tx = 0.$$

This shows that $V = R(T) + N(T)$. By the Rank-Nullity theorem, we can therefore conclude that this sum is direct (see Prob. 2.1.35a). Also note that T acts as the identity on $R(T)$.

This is in fact a characterization of all maps T with the property $T^2 = T$. Precisely, given a direct sum $V = V_1 \oplus V_2$, any vector $v \in V$ is decomposed uniquely into a sum $v = v_1 + v_2$, $v_1 \in V_1$, $v_2 \in V_2$. By defining $Tv := v_1$, we will obtain a linear map satisfying $T^2 = T$ whose range is V_1, where it acts as the identity, and whose kernel is V_2.

Remark: Such maps are called *linear projectors.* As we just saw, they are completely characterized by their range, onto which they project, and their kernel, which they annihilate.

Prob 2.3.20. (a) There are no cliques in this relation. To see that, use the result of Problem 19. The matrix B corresponding to the given incidence matrix A is

$$B = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}, \qquad \text{so} \quad B^3 = \begin{bmatrix} 0 & 2 & 0 & 3 \\ 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 \\ 3 & 0 & 2 & 0 \end{bmatrix}.$$

Since the diagonal entries of B^3 are all zero, there are no cliques.

(b) The clique consists of vertices 1, 3 and 4. To see this, use the same method to obtain

$$B = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}, \qquad B^3 = \begin{bmatrix} 2 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 \\ 3 & 0 & 2 & 3 \\ 3 & 0 & 3 & 2 \end{bmatrix}.$$

Remark: Alternatively, these conclusions could be reached by simply drawing the graphs.

Prob 2.4.1. (a) False. The bases α and β must be interchanged, i.e., $([T]_\alpha^\beta)^{-1} = [T^{-1}]_\beta^\alpha$. (b) True. (c) False. Recall the diagram in class. T maps V to W, whereas L_A maps $\mathbb{F}^n$ to $\mathbb{F}^m$ where n is the dimension of V and m is the dimension of W. So, L_A is *not the same* map even though L_A *represents* T. (d) False. The dimensions do not match. (e) True, since that's when the dimensions match. (f) False. Take, for example,

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

(g) True. (h) True. (i) True.

Prob 2.4.2. (a) No, the dimensions of the domain and the target do not match. (b) No. Same reason as in (a). (c) Yes, the inverse is given by $T^{-1}(b_1, b_2, b_3) = ((b_3 - 4b_2)/3, b_2, (b_3 - b_1 - 4b_2)/2)$. Alternatively, it is easy to check that the kernel of T is trivial, i.e., T is $1-1$. Since the dimensions of the domain and the target match, this implies that T is onto as well, hence is invertible. (d) No. Same as in (a). (e) No. Same as in (a). (f) Yes. The dimensions agree and the kernel of T is trivial.

Prob 2.4.6. If A is invertible and $AB = 0$, then $A^{-1}(AB) = A^{-1}AB = B = 0$.

Prob 2.4.17. (a) Since V_0 is closed under linear combinations and since T is linear, the image $T(V_0)$ of V_0 is also close dunder linear combinations. (Note that we did not use the fact that T is an isomorphism here, only that it is a linear map.)

(b) Let $\alpha = \{u_1, \ldots, u_n\}$ be a basis for V_0. Let us show that $T\beta = \{Tu_1, \ldots, Tu_n\}$ is a basis for $T(V_0)$. We already know $T\beta$ spans $T(V_0)$, so we just need to check that the set $T\beta$ is linearly independent. Indeed, since T is an isomorphism, its kernel is trivial, hence no nontrivi

Prob 2.4.22. The dimensions of $P_n(\mathbb{F})$ and $\mathbb{F}^{n+1}$ match, so it is enough to prove that the kernel of T is trivial. Use the Lagrange polynomials $f_j(x)$, $j = 0, \ldots, n$, associated with $c_0, c_1, \ldots, c_n$. By the Lagrange interpolation formula, any polynomial f can be represented in the form

$$f(x) = \sum_{j=0}^{n} f(c_j) f_j(x).$$

So, if $f(c_j) = 0$ for all $j = 0, \ldots, n$, this implies $f = 0$. Thus $N(T) = \{0\}$ and we are done.