Solutions to Homework 5.

Math 110, Fall 2006.

Prob 2.5.3.

$$(\mathbf{a})\ \begin{bmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_0 & b_0 & c_0 \end{bmatrix}, \quad (\mathbf{b})\ \begin{bmatrix} a_0 & b_0 & c_0 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{bmatrix}, \quad (\mathbf{c})\ \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ -3 & 2 & 1 \end{bmatrix},$$
$$(\mathbf{d})\ \begin{bmatrix} 2 & 1 & 1 \\ 3 & -2 & 1 \\ -1 & 3 & 1 \end{bmatrix}, \quad (\mathbf{e})\ \begin{bmatrix} 5 & -6 & 3 \\ 0 & 4 & -1 \\ 3 & -1 & 2 \end{bmatrix}, \quad (\mathbf{f})\ \begin{bmatrix} -2 & 1 & 2 \\ 3 & 4 & 1 \\ -1 & 5 & 2 \end{bmatrix}.$$

Prob 2.5.6. The matrix Q is simply the change of basis matrix where the change is from the standard basis to the basis β. That is, Q is the matrix whose columns are vectors in β.

(a)

$$[L_A]_\beta = \begin{bmatrix} 6 & 11 \\ -2 & -4 \end{bmatrix}, \qquad Q = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}.$$

(b)

$$[L_A]_\beta = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}, \qquad Q = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

(c)

$$[L_A]_\beta = \begin{bmatrix} 2 & 2 & 2 \\ -2 & -3 & -4 \\ 1 & 1 & 2 \end{bmatrix}, \qquad Q = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 2 \end{bmatrix}.$$

(d)

$$[L_A]_\beta = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 12 & 0 \\ 0 & 0 & 18 \end{bmatrix}, \qquad Q = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ -2 & 0 & 1 \end{bmatrix}.$$

Prob 2.5.7.

(a) By the linearity of T, $T(x, y) = (ax + by, cx + dy)$ for some a, b, c, d. To find these numbers, notice that every vector on the line remains unchanged under the action of T, i.e., $(ax + bmx, cx + dmx) = T(x, mx) = (x, mx)$ for all x. This implies $a + bm = 1$, $c + dm = m$. Now, every vector perpendicular to the line gets reflected about the line, i.e., $(amx - bx, cmx - dx) = T(mx, -x) = (-mx, x)$ for all x, which implies $am - b = -m$, $cm - d = 1$. Solving for a, b, c, d, we get

$$a = \frac{1 - m^2}{1 + m^2}, \quad b = c = \frac{2m}{1 + m^2}, \quad d = \frac{m^2 - 1}{1 + m^2}.$$

So, $T(x, y) = ((1 - m^2)x + 2my, 2mx + (m^2 - 1)y)/(1 + m^2)$.

(b) The same approach works for the projection P. We must have $P(x, mx) = (x, mx)$ and $P(-mx, x) = (0, 0)$. This gives $P(x, y) = (x + my, m(x + my))/(1 + m^2)$.

Prob 2.5.10. Suppose A and B are similar. Then there exists an invertible matrix Q such that $B = Q^{-1}AQ$. Since $\text{tr}(AB) = \text{tr}(BA)$ for all matrices A, B, we obtain:

$$\text{tr}(B) = \text{tr}(Q^{-1}AQ) = \text{tr}((Q^{-1}A)Q) = \text{tr}(Q(Q^{-1}A)) = \text{tr}(QQ^{-1}A) = \text{tr}(A).$$

Prob 2.6.1. **(a)** False, the target has to be the ground field. **(b)** True, since both the domain and the target are 1-dimensional. **(c)** True, since they have the same dimension. **(d)** True, $V = (V^*)^*$ for any finite-dimensional vector space V. **(e)** False, T may be an isomorphism different from the standard that maps β to β^*. **(f)** True, directly from the definition. **(g)** True, if a basis β of V is in $1-1$ correspondence with a basis γ of W, then the basis β^* of V^* is in $1-1$ correspondence with γ^* of W^*. **(h)** False, the derivative is a function, i.e., the target of the differentiation map is not $\mathbb{R}$.

Prob 2.6.2.

(a) Yes, since differentiation and evaluation at a fixed point are linear, and since the target of the map $\mathbf{f}$ is the ground field $\mathbb{R}$.

(b) No: the target of $\mathbf{f}$ is two-dimensional.

(c) Yes: tr is linear and maps a matrix into the ground field $\mathbb{F}$.

(d) No: the function is quadratic, not linear.

(e) Yes: definite integration is linear and maps into $\mathbb{R}$.

(f) Yes: Taking one of the elements of A is a linear operation that maps into $\mathbb{F}$.

Prob 2.6.4. Let A be the matrix representing $\{\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3\}$, i.e., the matrix whose rows are coordinates of the vectors $\mathbf{f}_j$ in the standard basis of V^*

$$\mathbf{e}_1(x,y,z) = x, \quad \mathbf{e}_2(x,y,z) = y, \quad \mathbf{e}_3(x,y,z) = z:$$

$$A = \begin{bmatrix} 1 & -2 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & -3 \end{bmatrix}.$$

Since the matrix A is invertible, the collection $\{\mathbf{f}_j : j = 1,2,3\}$ is a basis of V^*. The dual basis is

$$u_1 = \begin{bmatrix} 0.4 \\ -0.3 \\ -0.1 \end{bmatrix}, \quad u_2 = \begin{bmatrix} 0.6 \\ 0.3 \\ 0.1 \end{bmatrix}, \quad u_3 = \begin{bmatrix} 0.2 \\ 0.1 \\ -0.3 \end{bmatrix}.$$

It can be read off the columns of the matrix A^{-1}.

Prob 2.6.6. **(a)** $(T^t\mathbf{f})(x,y) = 7x + 4y$.

(b) and **(c)** $$[T]_\beta = \begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix}, \quad [T^t]_{\beta^*} = ([T]_\beta)^t = \begin{bmatrix} 3 & 1 \\ 2 & 0 \end{bmatrix}.$$

Prob 2.6.14. Let $\{x_1, \ldots, x_k\}$ be an ordered basis of W. Extend it to a basis $\beta = \{x_1, \ldots, x_n\}$ of V and form its dual basis $\beta^* = \{\mathbf{f}_1, \ldots, \mathbf{f}_n\}$. Let us prove that $\{\mathbf{f}_{k+1}, \ldots, \mathbf{f}_n\}$ is a basis of W^0.

Indeed, each function $\mathbf{f}_j$, $j > k$ annihilates each x_i for $i \leq k$, hence it annihilates their entire span W. So, all functionals $\mathbf{f}_j$, $j = k+1, \ldots, n$, are in W^0. Also, by Theorem 2.24, any function $\mathbf{f} \in V^*$ can be written as $\mathbf{f} = \sum_{j=1}^n \mathbf{f}(x_j)\mathbf{f}_j$. If $\mathbf{f} \in W^0$, then $\mathbf{f}(x_i) = 0$ for all $i = 1, \ldots, k$, so $\mathbf{f} \in \text{span}\{\mathbf{f}_j : j = k+1, \ldots, n\}$. Thus,

$\text{span}\{\mathbf{f}_j : j = k+1, \ldots, n\} = W^0$. Finally, the vectors $\{\mathbf{f}_j\}$ are linearly independent as part of a basis of V^*. Hence they form a basis for W^0.

Prob 2.6.15. A vector $\mathbf{f} \in W^*$ is in $N(T^t)$ if and only if $T^t\mathbf{f} = 0$, which is the same as $\mathbf{f}T = 0$, i.e., $\mathbf{f}Tx = 0$ for all $x \in V$. In other word, the linear functional $\mathbf{f}$ annihilates every vector of the form Tx, i.e., every vector in the range $R(T)$. So, $N(T^t) = (R(T))^0$.