Solutions to Homework 7.

Math 110, Fall 2006.

Prob 3.1.3. To find the inverses of the elementary matrices, we must simply undo the corresponding elementary operations:

$$(\mathbf{a})\quad \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \qquad (\mathbf{b})\quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad (\mathbf{c})\quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}.$$

Prob 3.1.5. If E is an elementary matrix of type I corresponding to the swap or rows i and J, then E has 1 in positions (i, j) and (j, i), zeros in positions (i, i) and (j, j), and the other entries exactly as in the identity matrix of the appropriate size. But then E is symmetric, i.e., its transpose is E itself. Likewise, if E is of type II, then its transpose is itself. If E is of type III and corresponds to adding row j multiplied by c to row i, then it looks like the identity matrix except for the entry c in position (i, j). Then its transpose has c in position (j, i), which corresponds to adding row i multiplied by c to row j.

So, in all three cases E is an elementary matrix if and only if so is E^t.

Prob 3.2.2. (a) 2, (b) 3, (c) 2, (d) 1, (e) 3, (f) 3, (g) 1.

Prob 3.2.6.

(a) T is invertible. Writing the matrix of T in the standard basis for $P_2(\mathbb{R})$ and inverting it, we get

$$T^{-1}(ax^2 + bx + c) = -ax^2 - (4a + b)x - (10a + 2b + c).$$

(b) T is not invertible, since it has a nontrivial kernel, for example, $T1 = 0$.

(c) T is invertible: the same approach as in (a) yields

$$T^{-1}(a, b, c) = \left(\frac{1}{6}a - \frac{1}{3}b + \frac{1}{2}c, \frac{1}{2}a - \frac{1}{2}c, -\frac{1}{6}a + \frac{1}{3}b + \frac{1}{2}c\right).$$

(d) Representing T with respect to the standard bases for $\mathbb{R}^3$ and $P_2(\mathbb{R})$ and inverting the obtained matrix, we get

$$T^{-1}(a + bx + cx^2) = (c, (a - b)/2, (a + b - 2c)/2).$$

(e) T is invertible, and

$$T^{-1}(a, b, c) = \left(\frac{1}{2}a - b + \frac{1}{2}c\right)x^2 + \left(-\frac{1}{2}a + \frac{1}{2}c\right)x + b.$$

(f) T is not invertible: for example, the matrix

$$A = \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}$$

is in the kernel of T, since both $\text{tr}(A)$ and $\text{tr}(EA)$ are zero, and the other two traces are also zero since $\text{tr}(A^t) = \text{tr}(A)$ and $\text{tr}(EA) = \text{tr}(AE)$ always.

Prob 3.2.8. Since $cA = (cI)A$ and since the matrix cI is invertible for $c \neq 0$, we conclude that multiplication by cI does not change the rank of A, i.e., $\text{rank}(A) = \text{rank}(cA)$.

Prob 3.2.18. Let $A_1, \ldots, A_n$ be the columns of A and let $B_1, \ldots, B_n$ be the rows of B. Then

$$AB = \begin{bmatrix} A_1 & \cdots & A_n \end{bmatrix} \cdot \begin{bmatrix} B_1 \\ \vdots \\ B_n \end{bmatrix} = A_1B_1 + \cdots + A_nB_n.$$

Each matrix A_jB_j is a product of a column-vector and a row-vector, hence has rank at most 1. So, AB is a sum of n matrices each of which has rank at most 1.

Prob 3.3.1. (a) False, an inhomogeneous system may not have any solutions. (b) False, it may have multiple solutions. (c) True, the zero vector is always a solution. (d) False if the matrix of the system is not invertible. (e) False for inhomogeneous systems (as above). (f) False; the homogeneous system always has a solution but the inhomogeneous system may not. (g) True, the solution (zero) is then unique. (h) False for inhomogeneous system: it is a shifted subspace.

Prob 3.3.11. The farmer, tailor, and carpenter must have incomes in the proportions $4 : 3 : 4$ since the vector $[434]^t$ is the eigenvector of A corresponding to its eigenvalue 1.

Prob 3.4.1. (a) False since operations on columns do not preserve the solutions. (b) True, row operations preserve the solutions. (c) True (proved in class). (d) True (proved in class). (e) False, it may or not be consistent depending on the vector in the right-hand side. (f) True (follows from properties of row echelon form). (g) True (like in (f)).

Prob 3.4.2.

(a) $x_1 = 4$, $x_2 = -3$, $x_3 = -1$. The reduced row echelon form of the augmented matrix is

$$\begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & -1 \end{bmatrix}.$$

(b) $x_1 = -5x_3 + 9$, $x_2 = -3x_3 + 4$, where $x_3 \in \mathbb{R}$ is free. The reduced row echelon form of the augmented matrix is

$$\begin{bmatrix} 1 & 0 & 5 & 9 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

(c) $x_1 = 2$, $x_2 = 3$, $x_3 = -2$, $x_4 = -1$. The reduced row echelon form of the augmented matrix is

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}.$$

(d) $x_1 = 7/13$, $x_2 = 16/13$, $x_3 = 14/13$, $x_4 = -18/13$. The reduced row echelon form of the augmented matrix is

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 7/13 \\ 0 & 1 & 0 & 0 & 16/13 \\ 0 & 0 & 1 & 0 & 14/13 \\ 0 & 0 & 0 & 1 & -18/13 \end{bmatrix}.$$

(e) $x_1 = 4x_2 + x_4 + 4$, $x_3 = 2x_4 + 1$, where x_2, $x_4 \in \mathbb{R}$ are free. The reduced row echelon form of the augmented matrix is

$$\begin{bmatrix} 1 & -4 & 0 & -1 & 4 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

(f) $x_1 = x_4 - 3$, $x_2 = 3$, $x_2 = -2x_4 + 3$, $x_3 = 1$, where $x_4 \in \mathbb{R}$ is free. The reduced row echelon form of the augmented matrix is

$$\begin{bmatrix} 1 & 0 & 0 & -1 & -3 \\ 0 & 1 & 0 & 2 & 3 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix}.$$

(g) $x_1 = x_2 - 23x_5 - 23$, $x_3 = 6x_5 + 7$, $x_4 = 9x_5 + 9$, where x_2, $x_5 \in \mathbb{R}$ are free. The reduced row echelon form of the augmented matrix is

$$\begin{bmatrix} 1 & -1 & 0 & 0 & 23 & -23 \\ 0 & 0 & 1 & 0 & -6 & 7 \\ 0 & 0 & 0 & 1 & -9 & 9 \end{bmatrix}.$$

(h) $x_1 = -x_3 - 2x_5 + 3$, $x_2 = -2x_3 - 5x_5 + 7$, $x_4 = x_5 - 3$, where x_3, $x_5 \in \mathbb{R}$ are free. The reduced row echelon form of the augmented matrix is

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 2 & 3 \\ 0 & 1 & 2 & 0 & 5 & 7 \\ 0 & 0 & 0 & 1 & -1 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

(i) $x_1 = x_5 + 2$, $x_2 = 2x_3 - 4x_5$, $x_4 = -2x_5 - 1$, where x_3, $x_5 \in \mathbb{R}$ are free. The reduced row echelon form of the augmented matrix is

$$\begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 2 \\ 0 & 1 & -2 & 0 & 4 & 0 \\ 0 & 0 & 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

(j) $x_1 = 2x_5 + 1$, $x_2 = 3x_5$, $x_3 = 1$, $x_4 = 2x_5 - 1$, where $x_5 \in \mathbb{R}$ is free. The reduced row echelon form of the augmented matrix is

$$\begin{bmatrix} 1 & 0 & 0 & 0 & -2 & 1 \\ 0 & 1 & 0 & 0 & -3 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & -2 & -1 \end{bmatrix}.$$