Solutions to Homework 9.

Math 110, Fall 2006.

Prob 4.3.10. Since $\det(AB) = \det(A)\det(B)$ for any two square matrices of the same order, this implies, by induction, that $(\det M)^k = \det(M^k)$ for all $k \in \mathbb{N}$. Since the determinant of the zero matrix is zero, this shows that, for a nilpotent matrix M, $(\det M)^k = 0$, hence $\det M = 0$.

Prob 4.3.11. By Theorem 4.8, $\det M^t = \det M$. Now, if M is skew-symmetric, then

$$\det M = \det M^t = \det(-M) = (-1)^n \det M,$$

the last equality by the n-linearity of the determinant. If n is odd, this gives $\det M = -\det M$, i.e., $\det M = 0$. If n is even, it does not imply anything.

Prob 4.3.14. Since β consists of n vectors, β is a basis if and only if these vectors are linearly independent, which is equivalent to the map L_B being one-to-one. Since the matrix B is square, this is in turn equivalent to B being invertible, hence having a nonzero determinant. Thus, β is a basis if and only if $\det B \neq 0$.

Prob 4.3.15. If matrices A and B are similar, then, for some invertible matrix Q, we have $A = QBQ^{-1}$. Then, by Theorem 4.7 and its Corollary,

$$\det A = \det(QBQ^{-1}) = \det Q \cdot \det B \cdot \det Q^{-1} = \det Q \cdot \det B \cdot \frac{1}{\det Q} = \det B.$$

Prob 4.3.23. We first prove slightly different versions of (a) and (b) and then combine them to prove (a) and (b).

(a') Suppose that a $k\times k$ submatrix of A has a nonzero determinant. Then $\operatorname{rank}(A) \geq k$. Indeed, if there exists a $k\times k$ invertible submatrix of A, then its rows are linearly independent, hence the same rows of A are also independent, hence the rank of A is at least k.

(b') Suppose $\operatorname{rank}(A) \geq k$. Then there exists a $k\times k$ submatrix with a nonzero determinant. Indeed, since $\operatorname{rank}(A) \geq k$, the matrix A has k linearly independent rows. Let B be the matrix obtained from A by taking only those rows. Then $\operatorname{rank}(B) = k$. Since the column rank and the row rank of B are the same, there exist k linearly independent columns of B. Taking only those columns, we now obtain a $k\times k$ submatrix of A with linearly independent columns. Since this submatrix is square and its columns are linearly independent, it is invertible, hence its determinant is nonzero.

Now, (b') implies (b). To prove (a), take the largest integer such that A has a $k\times k$ invertible submatrix. Then, by (a'), $\operatorname{rank}(A) \geq k$. If $\operatorname{rank}(A) \geq k+1$, then, by (b'), there exists an invertible submatrix of A of size $(k+1)\times(k+1)$, contrary to the maximality of k. This proves (a).

Prob 4.4.5. Let the order of the identity block be k. Expanding $\det M$ by its last row, we notice that the last entry, which is equal to 1, is the only nonzero entry in that row. Hence

$$\det M = \det \begin{bmatrix} A & \widetilde{B} \\ 0 & I_{k-1} \end{bmatrix},$$

where the submatrix $\widetilde{B}$ is the submatrix B with its last column deleted, and I_{k-1} is the identity matrix of order $k-1$. Doing this $k-1$ more times, we obtain $\det M = \det A$.

Prob 4.5.1. (a) False, $\delta(tA) = t^n\delta(A)$, so this function is homogeneous of degree n rather than 1. (b) True (directly from the definition of n-linearity). (c) True (by Theorem 4.10). (d) False, $\delta(B) = -\delta(A)$. (e) False, it is unique up to a (scalar) normalization factor. (f) True, the zero function trivially satisfies the multilinearity property and the alternation property.

Prob 4.5.4. No. Multiply the first row of an arbitrary matrix A with $A_{22} \neq 0$ by a scalar $t \neq 1$ without changing the second and third row to produce a matrix B. Then $\delta(B) = A_{22} \neq t\delta(A)$.

Prob 4.5.6. No. Multiply the first row of an arbitrary matrix A with $A_{23} \neq 0$, $A_{32} = 0$ by a scalar $t \neq 1$ without changing the second and third row to produce a matrix B. Then $\delta(B) = tA_{11} + A_{23} + A_{32} \neq t(A_{11} + A_{23} + A_{32}) = t\delta(A)$.

Prob 4.5.10. Yes. This is a 3-linear function, since both products involve exactly one entry from each row.