MATH 110, mock final test.

Name
Student ID #

All the necessary work to justify an answer and all the necessary steps of a proof must be shown clearly to obtain full credit. Partial credit **may** be given but only for significant progress towards a solution. Show all relevant work in logical sequence and indicate all answers clearly. Cross out all work you do not wish considered. Books and notes are allowed. Calculators, computers, cell phones, pagers and other electronic devices are not allowed during the test.

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1. Using reduced row echelon form, determine whether the linear system

$$\begin{aligned} x_1 + 2x_2 - x_3 &= 1 \\ 2x_1 + x_2 + 2x_3 &= 3 \\ x_1 - 4x_2 + 7x_3 &= 4 \end{aligned}$$

has a solution.

2. Let A and B be linear maps on a vector space V such that $AB = 0$. Prove that $\operatorname{rank}(A) + \operatorname{rank}(B) \leq \dim V$.

3. Let A be an $n \times n$ matrix such that $A(i,j) = 0$ for more than $n^2 - n$ pairs of values of i and j. Prove that $\det A = 0$.

4. Squares are labelled 1 through 4 consecutively from left to right. A player begins by placing a marker in square 2. A die is rolled and the marker is moved one square to the left if 1 or 2 is rolled or one square to the right if 3, 4, 5 or 6 is rolled. This process continues until the marker ends in square 1 (winning the game) or in square 4 (losing the game). What is the probability of winning?

5. Find the general solution to the following system of differential equations:

$$\begin{aligned} x_1' &= 8x_1 + 10x_2 \\ x_2' &= -5x_1 - 7x_2 \end{aligned}$$

6. Let V be the vector space of polynomials in x and y of (total) degree at most 2, and let

$T : V \to V$ be a linear map defined by

$$(Tf)(x, y) = \frac{\partial}{\partial x} f(x, y) + \frac{\partial}{\partial y} f(x, y).$$

Find the Jordan canonical form of T, the corresponding Jordan basis, and the minimal polynomial of T.

7. Let $A : V \to V$ be a linear map on an n-dimensional vector space V.

- Prove that the set of all linear maps B on V satisfying the condition $AB = 0$ is a subspace of the space of all linear maps on V.
- Can every subspace of the space of all linear transformations on V be obtained in that manner, by the choice of a suitable A?
- What is the dimension of that subspace when A is a Jordan block of order n with eigenvalue 0?

8. Using the Gram-Schmidt procedure, find an orthonormal basis for the real vector space $\mathrm{span}\{\sin t, \cos t, 1\}$ equipped with the inner produt $\langle f, g \rangle = \int_0^\pi f(t)g(t)dt$.

9. Let T be an inner product space, and let y, $z \in V$. Define $T : V \to V$ by $Tx := \langle x, y \rangle z$. Prove that T is a linear map and find an explicit expression for T^*.

10. Let V be the inner product space of complex-valued continuous functions on $[0, 1]$ with the inner product

$$\langle f, g \rangle = \int_0^1 f(t)\overline{g(t)}dt.$$

Let $h \in V$, and define $T : V \to V$ by $Tf := hf$. Prove that T is a unitary operator if and only if $|h(t)| = 1$ for all $t \in [0, 1]$.