# MATH 110, solutions to the mock midterm.

1. Consider the vector space $P(\mathbb{R})$ and the subsets $V$ consisting of those vectors (polynomials) $f$ for which:

**(a)** $f$ has degree 3,

**(b)** $2f(0) = f(1)$,

**(c)** $f(t) \geq 0$ whenever $t \geq 0$,

**(d)** $f(t) = f(1-t)$ for all $t$.

In which of these cases is $V$ a subspace of $P(\mathbb{R})$?

**Solution.** (a) This is not a subspace of $P(\mathbb{R})$. Indeed, adding or subtracting two polynomials of exact degree 3 may result in a polynomial of a smaller degree, e.g., if $f(x) = x^3 - 3x$ and $g(x) = x^3$, then $g(x) - g(x) = 3x$, and $\deg(g-f) = 1$. (Another observation that shows that $V$ is not a subspace is that the zero polynomial 0 is not in $V$.)

(b) This is a subspace of $P(\mathbb{R})$. The zero polynomial satisfies the defining condition $2f(0) = f(1)$. Also, if $2f(0) = f(1)$ and $2g(0) = g(1)$, then

$$2(\alpha f + \beta g)(0) = \alpha f(0) + \beta g(0) = \alpha f(1) + \beta g(1) = (\alpha f + \beta g)(1)$$

for all $\alpha$, $\beta \in \mathbb{R}$. So $V$ is a subspace of $P(\mathbb{R})$.

(c) This is not a subspace, the (additive) inverse of a function that is nonnegative over $\mathbb{R}_+$ is nonpositive over $\mathbb{R}_+$. E.g., if $f(x) = x^2$, then $f \in V$, but $-f \notin V$. Thus $V$ is not a subspace of $P(\mathbb{R})$.

(d) This is a subspace of $P(\mathbb{R})$. Indeed, the zero polynomial satisfies the condition $f(t) = f(1-t)$. If two functions, $f$ and $g$ satisfy this condition, then so are all their linear combinations $\alpha f + \beta g$, i.e.,

$$(\alpha f + \beta g)(t) = \alpha f(t) + \beta g(t) = \alpha f(1-t) + \beta g(1-t) = (\alpha f + \beta g)(1-t).$$

So $V$ is a subspace of $P(\mathbb{R})$.

2.

$$\text{Let } A = \begin{bmatrix} 0 & 2 & 3 \\ -1 & 3 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} -2 & 0 \\ 6 & 4 \\ -4 & 6 \end{bmatrix}, \quad v = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}, \quad w = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}.$$

Do the products $Aw$, $B^t v^t$, $vAw$ exist? Evaluate those that do. Is the set $\{A, B^t\}$ linearly independent?

**Solution.** The matrix $A$ is of size $2\times 3$ whereas $w$ is of size $3\times 1$. Since the inner dimensions agree, the product $Aw$ exists. The sizes of the pair $B^t$, $v^t$ are exactly the same, so the product $B^t v^t$ exists as well. Now, the product $Aw$ is of size $2\times 1$, but the size of $v$ is $3\times 1$, so the inner dimensions 1 and 2 disagree, and the product $vAw = v(Aw)$ does not exist. The products $Aw$ and $B^t v^t$ are equal to

$$Aw = \begin{bmatrix} 4 \\ -7 \end{bmatrix}, \qquad B^t v^t = (vB)^t = \begin{bmatrix} -2 & 26 \end{bmatrix}^t = \begin{bmatrix} -2 \\ 26 \end{bmatrix}.$$

The set $\{A, B^t\}$ is linearly independent: Assume $\alpha A + \beta B^t = 0_{2\times 3}$, i.e.,

$$\alpha \begin{bmatrix} 0 & 2 & 3 \\ -1 & 3 & -2 \end{bmatrix} + \beta \begin{bmatrix} -2 & 6 & 4 \\ 0 & 4 & 6 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Since the $(1,1)$ entry is zero, we conclude that $\beta = 0$, and since the $(2,1)$ entry is zero, we get $\alpha = 0$. So, the matrices $A$ and $B^t$ are linearly independent.

3.

$$\text{Let } A = \begin{bmatrix} 0 & 2 \\ 2 & -2 \end{bmatrix}, \quad \beta = \{\begin{bmatrix} -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\}.$$

Find the representation $[L_A]_\beta$, the dual basis $\beta^*$, and the matrix $[(L_A)^t]_{\beta^*}$.

**Solution.** The map $L_A$ acts on $\beta$ as follows :

$$L_A\beta = \{\begin{bmatrix} 2 \\ -4 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix}\}.$$

Since

$$\begin{bmatrix} 2 \\ -4 \end{bmatrix} = -3 \begin{bmatrix} -1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad \begin{bmatrix} 2 \\ 0 \end{bmatrix} = - \begin{bmatrix} -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

we conclude

$$[L_A]_\beta = \begin{bmatrix} -3 & -1 \\ -1 & 1 \end{bmatrix}.$$

The dual basis for $\beta$ is

$$\mathbf{f}_1(x,y) = -x/2 + y/2, \qquad \mathbf{f}_2(x,y) = x/2 + y/2,$$

and by the theorem on duals of linear maps,

$$[(L_A)^t]_\beta = \begin{bmatrix} -3 & -1 \\ -1 & 1 \end{bmatrix}.$$

4. Let

$$A \; : \; P_n(\mathbb{R}) \to P_n(\mathbb{R}) \quad : \; (Af)(t) := f(t+1).$$

Prove that

$$A = I + \frac{D}{1!} + \frac{D^2}{2!} + \cdots + \frac{D^n}{n!},$$

where $D$ is the differentiation operator on $P_n(\mathbb{R})$.

**Proof.** Since any polynomial is infinitely differentiable, we can apply Taylor's formula of any order to expand $f(t+1)$ around the point $t$. By using Taylor's formula of order $n$, we obtain

$$f(t+1) = f(t) + \frac{f'(t)}{1!} + \frac{f''(t)}{2!} + \cdots + \frac{f^{(n)}}{n!} + \frac{f^{(n+1)}(\xi)}{(n+1)!}, \tag{1}$$

where $\xi \in (t, t+1)$. But it $f$ is a polynomial of degree $n$ or smaller, then $f^{(n+1)}(\xi) = 0$ for any point $\xi \in \mathbb{R}$. So the last term in (1) disappears and we see that

$$Af = (I + \frac{D}{1!} + \cdots + \frac{D^n}{n!})f \quad \text{for all } f \in P_n(\mathbb{R}). \qquad \square$$

5. Let $m < n$ and let $\mathbf{f}_1, \ldots, \mathbf{f}_m$ be linear functionals on an $n$-dimensional space $V$. Prove that there exists a nonzero vector $x \in V$ such that $\mathbf{f}_j x = 0$ for all $j = 1, \ldots, m$. What does this result say about solutions of linear equations?

**Proof.** Without loss of generality we may assume that the set $\{\mathbf{f}_j : j = 1, \ldots, m\}$ is linearly independent, since the result for linearly dependent functionals will follow from the result for linearly independent functionals.

So, assuming independence, complete $\{\mathbf{f}_j : j = 1, \ldots, m\}$ to a basis $\{\mathbf{f}_j : j = 1, \ldots, n\}$ of $V^*$, and consider its dual basis $\{u_j : j = 1, \ldots, n\}$ of $V$. Since $\mathbf{f}_j u_i = \delta_{ij}$, we see that a nonzero vector $x := u_n$ satisfies the required condition $\mathbf{f}_j x = 0$ for all $j = 1, \ldots, m$. $\square$

Since the action of a linear functional on $\mathbb{R}^n$ is realized as multiplication from the left by a row vector, this result says the following: Any linear homogeneous system with fewer equations than unknowns has a nontrivial solution.

6. Reduce the matrix

$$\begin{bmatrix} 1 & -1 & 4 & 3 & -2 & -2 \\ 0 & 2 & 0 & 1 & 1 & 3 \\ -1 & 3 & -4 & -2 & 3 & 5 \end{bmatrix}$$

to its reduced row echelon form. Show all steps.

**Solution.** To create 0 in position $(3,1)$, add the first row to the third row. This gives

$$\begin{bmatrix} 1 & -1 & 4 & 3 & -2 & -2 \\ 0 & 2 & 0 & 1 & 1 & 3 \\ 0 & 2 & 0 & 1 & 1 & 3 \end{bmatrix}.$$

To create zero in position $(3,2)$, subtract the second row from the third to get

$$\begin{bmatrix} 1 & -1 & 4 & 3 & -2 & -2 \\ 0 & 2 & 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

To make the $(2,2)$ entry equal to 1, divide the second row by 2 to get

$$\begin{bmatrix} 1 & -1 & 4 & 3 & -2 & -2 \\ 0 & 1 & 0 & 1/2 & 1/2 & 3/2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Finally, to create 0 in position $(1,2)$, add the second row to the first row to obtain

$$\begin{bmatrix} 1 & 0 & 4 & 7/2 & -3/2 & -1/2 \\ 0 & 1 & 0 & 1/2 & 1/2 & 3/2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

This matrix is in reduced row echelon form.