

Solutions to the MATH 118 midterm test.

1. (10pts total; 5 pts each subitem) Suppose $K(x, y)$ is a continuous compactly supported function on $\mathbb{R} \times \mathbb{R}$. Define $T : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$ by

$$T(f)(x) := \int_{y \in \mathbb{R}} f(y)K(x, y) dy.$$

Show that T is a linear map and that its adjoint is given by

$$T^*(g)(x) := \int_{y \in \mathbb{R}} g(y)\overline{K(y, x)} dy.$$

Solution. The map T is linear, since it respects linear combinations of functions:

$$\begin{aligned} T(\alpha f + \beta g)(x) &= \int_{y \in \mathbb{R}} (\alpha f(y) + \beta g(y))K(x, y) dy = \alpha \int_{y \in \mathbb{R}} f(y)K(x, y) dy \\ &\quad + \beta \int_{y \in \mathbb{R}} g(y)K(x, y) dy = \alpha(Tf)(x) + \beta(Tg)(x). \end{aligned}$$

To find the adjoint, we form the inner product of Tf and g (where both f and g are arbitrary), change the order of integration, and rename variables of integration:

$$\begin{aligned} \langle Tf, g \rangle &= \int_{x \in \mathbb{R}} \int_{y \in \mathbb{R}} f(y)K(x, y) dy \overline{g(x)} dx = \int_{y \in \mathbb{R}} f(y) \int_{x \in \mathbb{R}} \overline{K(x, y)}g(x) dx \\ &= \int_{x \in \mathbb{R}} f(x) \int_{y \in \mathbb{R}} \overline{K(y, x)}g(y) dy dx = \langle f, T^*g \rangle. \end{aligned}$$

Since f and g are arbitrary, this implies that $(T^*g)(x) = \int_{y \in \mathbb{R}} \overline{K(y, x)}g(y) dy$.

2. (10pts total) Show that the sinc function

$$\text{sinc}(x) := \begin{cases} \frac{\sin(\pi x)}{\pi x} & x \neq 0 \\ 1 & x = 0 \end{cases}$$

solves the refinement equation

$$\phi(x) = \phi(2x) + \sum_{k \in \mathbb{Z}} \frac{2(-1)^k}{(2k+1)\pi} \phi(2k - 2k - 1).$$

Solution. Recall that the Fourier transform of the characteristic function $\chi_{[-\pi, \pi]}$ of the interval $[-\pi, \pi]$ is (almost) the sinc function:

$$\int_{-\pi}^{\pi} e^{-i\lambda x} dx = \frac{e^{-i\lambda x}}{-i\lambda} \Big|_{x=-\pi}^{x=\pi} = 2\pi \frac{\sin(\pi\lambda)}{\pi\lambda}.$$

Stretching this function and modulating it (multiplying by exponentials) will produce the desired shifts

$$\frac{1}{2} \int_{-2\pi}^{2\pi} e^{-i\lambda x + ikx/2} dx = 2\pi \frac{\sin(\pi(2\lambda - k))}{\pi(2\lambda - k)}.$$

Thus we need to solve the following equation on the Fourier domain:

$$\chi_{[-\pi, \pi]}(\lambda) = \chi_{[-2\pi, 2\pi]}(\lambda) \sum_{k \in \mathbb{Z}} \frac{p_k}{2} e^{ik\lambda/2}.$$

So, we must find the Fourier series for $\chi_{[-\pi, \pi]}$ on the interval $[-2\pi, 2\pi]$. Since

$$\int_{-\pi}^{\pi} e^{-ik\lambda/2} d\lambda = \begin{cases} 2\pi & k = 0 \\ \frac{4 \sin(k\pi/2)}{k} = \begin{cases} 0 & k \text{ is even} \\ \frac{4(-1)^{(k-1)/2}}{k} & k \text{ is odd} \end{cases} & k \neq 0, \end{cases}$$

we get

$$\chi_{[-\pi, \pi]}(\lambda) = \chi_{[-2\pi, 2\pi]}(\lambda) \left(\frac{1}{2} + \sum_{k \in \mathbb{Z}} \frac{(-1)^k}{2k+1} e^{i(2k+1)\lambda/2} \right),$$

hence

$$\phi(x) = \phi(2x) + \sum_{k \in \mathbb{Z}} \frac{2(-1)^k}{(2k+1)\pi} \phi(2x - 2k - 1).$$

3. (12pts total; 2pts for each subitem) Are the following filters (a) linear (b) time-invariant (c) causal?

1. $(L_1 f)(t) = f(t) - \int_t^{t^2} f(x) dx$,
2. $(L_2 f)(t) := \int_{-\infty}^{\infty} f(x) e^{-(x-t)^2} dx$.

Solution. The first filter is linear, not time-invariant, and not causal. Indeed, it is a linear combination of the identity map and an integration operator, and both are linear, so the filter is linear as well. It is not time-invariant, which can be seen by applying it to a shift f_a of an arbitrary function f :

$$\begin{aligned} (L_1 f_a)(t) &= f_a(t) - \int_t^{t^2} f_a(x) dx = f(t-a) - \int_t^{t^2} f(x-a) dx \\ &= f(t-a) - \int_{t-a}^{t^2-a} f(x) dx \neq f(t-a) - \int_{t-a}^{(t-a)^2} f(x) dx = (L_1 f)(t-a). \end{aligned}$$

L_1 is not causal, which can be checked, for example, by applying L_1 to the function f which is 0 over \mathbb{R}_- and 1 over \mathbb{R}_+ : for $t < 0$, we get

$$(L_1 f)(t) = - \int_0^{t^2} dx = -t^2 \neq 0.$$

The second filter is linear and time-invariant, since it is given explicitly as a convolution with the function e^{-x^2} . Since the support of this function is all of \mathbb{R} , the filter is not causal (causality requires that the support be a subset of \mathbb{R}_+).

4. (8pts total) Let ϕ and ψ be the Haar scaling function and the Haar wavelet, respectively. We denote, as usual,

$$V_j := \text{span}\{\phi(2^j \cdot -k), k \in \mathbb{Z}\}, \quad W_j := \text{span}\{\psi(2^j \cdot -k), k \in \mathbb{Z}\}.$$

Express the function f given by

$$f(x) := \begin{cases} 2 & 0 \leq x \leq 1/4, \\ -3 & 1/4 \leq x < 1/2, \\ 1 & 1/2 \leq x < 3/4, \\ 3 & 3/4 \leq x < 1, \\ 0 & \text{otherwise} \end{cases}$$

in terms of its components in V_0 , W_0 and W_1 .

Solution. We are given coefficients at level 2, i.e., $a^{[2]} = [2, -3, 1, 3]$. Using the wavelet decomposition algorithm for the Haar system, we get

$$\begin{aligned} a_0^{[1]} &= \frac{a_0^{[2]} + a_1^{[2]}}{2} = -\frac{1}{2}, & a_1^{[1]} &= \frac{a_2^{[2]} + a_3^{[2]}}{2} = 2, \\ b_0^{[1]} &= \frac{a_0^{[2]} - a_1^{[2]}}{2} = \frac{5}{2}, & b_1^{[1]} &= \frac{a_2^{[2]} - a_3^{[2]}}{2} = -1, \\ a_0^{[0]} &= \frac{a_0^{[1]} + a_1^{[1]}}{2} = \frac{3}{4}, & b_0^{[0]} &= \frac{a_0^{[1]} - a_1^{[1]}}{2} = -\frac{5}{4}. \end{aligned}$$

So, the representation of f in terms of its wavelet components from W_0 and W_1 and its coarse component from V_0 is

$$f(x) = \underbrace{\frac{3}{4}\phi(x)}_{\in V_0} - \underbrace{\frac{5}{4}\psi(x)}_{\in W_0} + \underbrace{\frac{5}{2}\psi(2x) - \psi(2x-1)}_{\in W_1}.$$