Solutions to homework # 1.

1. Define $\langle v, w\rangle$ for $v = (v_1, v_2)$ and $w = (w_1, w_2) \in \mathbb{C}^2$ as

$$\langle v, w\rangle = (\overline{w_1}, \overline{w_2}) \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}.$$

Does $\langle \cdot, \cdot\rangle$ define an inner product?

Solution. No, this does not define an inner product, since positivity fails. Consider the vector $v = (2, -1)$. While $v \neq 0$, we see that $\langle v, v\rangle = 0$.

2. For $n \in \mathbb{N}$, let

$$f_n(t) := \begin{cases} 1, & 0 \le t \le 1/n, \\ 0, & \text{otherwise.} \end{cases}$$

Show that $f_n \to 0$ in $L^2[0, 1]$. Show that f_n does not converge to zero uniformly on $[0, 1]$.

Solution. Since $\|f_n - 0\|^2 = \int_0^{1/n} dt = 1/n \to 0$ as $n \to \infty$, we see that f_n converges to 0 in $L^2[0, 1]$. On the other hand, $f_n(0) = 1$ for any n, hence f_n does not converge to 0 uniformly. (The pointwise limit of f_n is the function that is zero everywhere except at $x = 0$, where it is equal to 1. Still, f_n does not converge uniformly to that limit either.)

3. What is the orthogonal complement in $\mathbb{R}^3$ of the span of $(1, -2, 1)$?

Solution. Directly from the definition, it is the plane orthogonal to the vector $(1, -2, 1)$, i.e., the set

$$\{(x_1, x_2, x_3) : x_1 - 2x_2 + x_3 = 0\}.$$

This plane is the linear span of, say, $(1, 1, 1)$ and $(2, 1, 0)$.

4. Let $f(t) = 1$ on $[0, 1]$. What is the orthogonal complement of the span of f in $L^2[0, 1]$?

Solution. The span of f are all functions constant on $[0, 1]$. A function g is orthogonal to the span of f iff it is orthogonal to f itself. In other words, $\int_0^1 g(t)dt = 0$, i.e., the orthogonal complement in question is all functions in $L^2[0, 1]$ with mean value 0.

5. Suppose that f is a differentiable function that is orthogonal to cos in $L^2[0, \pi]$. Show that f' is orthogonal to sin in $L^2[0, \pi]$.

Solution. Using integration by parts, we get

$$0 = \int_0^\pi f(t)\cos(t)dt = f(t)\sin(t)|_0^\pi - \int_0^1 f'(t)\sin(t)dt = -\int_0^1 f'(t)\sin(t)dt.$$

Hence f' is orthogonal to sin.