Solutions to homework # 2.

1. By using Gram-Schmidt, find an orthonormal basis for the subspace of $L^2[0,1]$ spanned by 1, x, x^2, x^3.

Solution. The vector 1 is already normalized, so $e_1(x) = 1$. Since $\langle x, 1\rangle = 1/2$, x must be replaced by $x - 1/2$ and then normalized. As $\|x - 1/2\|_2 = 1/(2\sqrt{3})$, we obtain $e_2(x) = \sqrt{3}(2x-1)$. The vector x^2 then must be replaced by $x^2-x+1/6$ to make it orthogonal to 1 and x. Since $\|x^2-x+1/6\|_2 = 1/(6\sqrt{5})$, normalization gives $e_3(x) = \sqrt{5}(6x^2-6x+1)$. Finally, x^3 is replaced by $x^3-(3/2)x^2+(3/5)x-1/20$. Since $\|x^3-(3/2)x^2+(3/5)x-1/20\|_2 = 1/(20\sqrt{7})$, we get $e_4(x) = \sqrt{7}(20x^3 - 30x^2 + 12x - 1)$.

2. Find the $L^2[0,1]$-orthogonal projection of the function $\cos x$ onto the span of 1, x, x^2, x^3.

Solution. Denote

$$\begin{array}{rcccl} a_1 & := & \langle \cos x, 1\rangle & = & \sin 1, \\ a_2 & := & \langle \cos x, x\rangle & = & \sin 1 + \cos 1 - 1 \\ a_3 & := & \langle \cos x, x^2\rangle & = & 2\cos 1 - \sin 1 \\ a_4 & := & \langle \cos x, x^3\rangle & = & -5\sin 1 - 3\cos 1 + 6 \end{array}$$

Using the result of Problem 1, we see that the projection of $\cos x$ on the span of 1, x, x^2, x^3 is given by the formula

$$\sum_{j=1}^{4} \langle \cos, e_j\rangle e_j$$

with the functions e_j obtained in Problem 1. Hence the projection is the function

$$a_1 + \sqrt{3}(2a_2 - a_1)e_2(x) + \sqrt{5}(6a_3 - 6a_2 + a_1)e_3(x) + \sqrt{7}(20a_4 - 30a_3 + 12a_2 - a_1)e_4(x).$$

3. Suppose u_0 and u_1 are vectors in an inner product space V such that $\langle u_0, v\rangle = \langle u_1, v\rangle$ for all $v \in V$. Show that $u_0 = u_1$.

Solution. By the bilinearity of the product $\langle \cdot, \cdot \rangle$, we see that

$$\langle u_0 - u_1, v\rangle = 0 \qquad \text{for all} \quad v \in V.$$

Take $v := u_0 - u_1$ to conclude $\langle u_0 - u_1, u_0 - u_1\rangle = 0$. Now the positivity of $\langle \cdot, \cdot \rangle$ implies that $u_0 = u_1$.

4. Suppose A in an $n \times n$ matrix with complex entries. Show that the following are equivalent.

(a) The rows of A form an orthonormal basis in $\mathbb{C}^n$.

(b) $AA^* = I$ (the identity matrix).

(c) $\|Ax\| = \|x\|$ for all $x \in \mathbb{C}^n$.

Solution. $(a) \implies (b)$ The entries of the matrix AA^* are exactly the inner products of rows of A with themselves. Indeed, if

$$A = \begin{bmatrix} a_1 \\ \dots \\ a_n \end{bmatrix}$$

where the a_j's are rows of A, then $A^* = [a_1^*, \dots, a_n^*]$, where the a_j^*'s are conjugate transposes of the a_j's, so

$$AA^*(i,j) = a_i a_j^* = \delta_{ij} := \begin{cases} 1 & i = j, \\ 0 & i \neq j. \end{cases}$$

Thus $AA^* = I$.

$(b) \implies (c)$. If $AA^* = I$, then, in particular, A is a nonsingular matrix, and $A^{-1} = A^*$, hence $A^*A = I$ as well. Now, for all $x \in \mathbb{C}^n$,

$$\|x\|^2 = \langle x, x\rangle = \langle Ix, x\rangle = \langle A^*Ax, x\rangle = \langle Ax, Ax\rangle = \|Ax\|^2.$$

So $\|Ax\| = \|x\|$ for all $x \in \mathbb{C}^n$.

$(c) \implies (a)$. First let us check that the condition

$$\|Ax\| = \|x\| \qquad \text{for all } \ x \in \mathbb{C}^n$$

implies that

$$\langle Ax, Ay\rangle = \langle x, y\rangle \qquad \text{for all } \ x, \ y \in \mathbb{C}^n. \tag{1}$$

This fact is due to the polarization identity

$$\langle x, y\rangle = \frac{1}{4}\left(\langle x+y, x+y\rangle - \langle x-y, x-y\rangle + \mathrm{i}(\langle x+\mathrm{i}y, x+\mathrm{i}y\rangle - \langle x-\mathrm{i}y, x-\mathrm{i}y\rangle)\right).$$

Thus (1) holds. Then taking unit vectors e_i, e_j, we get $\langle Ae_i, Ae_j\rangle = \langle e_i, e_j\rangle = \delta_{ij}$. But the vectors Ae_j are columns of A. So, the columns of A are orthonormal. Hence $A^*A = I$ (as we saw before), so $AA^* = I$, hence the rows of A are also orthonormal.