Solutions to homework # 3.

1. Find the cosine expansion for x^2 on $[0,\pi]$.

Solution. For $n=0$, we get

$$a_0 = \frac{2}{\pi}\int_0^\pi x^2 \mathrm{dx} = \frac{2\pi^2}{3}.$$

For $n \geq 1$, we integrate by parts and get

$$a_n = \frac{2}{\pi}\int_0^\pi x^2\cos nx \mathrm{dx} = \frac{2}{\pi}\left(\frac{x^2\sin nx}{n} + \frac{2x\cos nx}{n^2} - \frac{2\sin nx}{n^3}\right)\Bigg|_0^\pi = \frac{4(-1)^n}{n^2}.$$

Thus,

$$x^2 = \frac{a_0}{2} + \sum_{n=1}^\infty a_n \cos nx = \frac{\pi^2}{3} + \sum_{n=1}^\infty \frac{4(-1)^n}{n^2}\cos nx.$$

This series converges to x^2 on $[-\pi,\pi]$ since x^2 is an even function.

2. Find the sine expansion for x^2 on $[0,\pi]$.

Solution. Using integration by parts, we get

$$\begin{aligned} b_n &= \frac{2}{\pi}\int_0^\pi x^2\sin nx\mathrm{dx} = \frac{2}{\pi}\left(-\frac{x^2\cos nx}{n} + \frac{2x\sin nx}{n^2} + \frac{2\cos nx}{n^3}\right)\Bigg|_0^\pi \\ &= \frac{2}{\pi}\left(\frac{(-1)^{n+1}\pi^2}{n} + \frac{2(-1)^n}{n^3} - \frac{2}{n^3}\right). \end{aligned}$$

Notice that the last two summands cancel each other for even terms. Hence

$$x^2 = \sum_{n=1}^\infty \frac{2\pi(-1)^{n+1}}{n}\sin nx - \sum_{n=1}^\infty \frac{8}{\pi(2n-1)^3}\sin(2n-1)x.$$

This series converges to $-x^2$ on the interval $[-\pi,0]$.

3. Find the Fourier series for $|\sin x|$ on $[-\pi,\pi]$.

Solution. Since the function $|sinx|$ is even, it is represented by a cosine series. We get

$$\begin{aligned} a_0 &= \frac{2}{\pi}\int_0^\pi \sin x\mathrm{dx} = \frac{4}{\pi}, \\ a_n &= \frac{2}{\pi}\int_0^\pi \sin x\cos nx\mathrm{dx} = \frac{2}{\pi}\cdot\frac{n^2}{n^2-1}\left(\frac{1}{n}\sin x\sin nx + \frac{1}{n^2}\cos x\cos nx\right)\Bigg|_0^\pi \\ &= \frac{2}{\pi}\cdot\frac{(-1)^{n+1}-1}{n^2-1}. \end{aligned}$$

Note that that $a_n = 0$ for all odd values of n. So,

$$|\sin x| = \frac{4}{\pi}\left(\frac{1}{2} - \sum_{n=1}^\infty \frac{\cos 2nx}{4n^2-1}\right).$$

4. Using Fourier series series, prove that

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{8}. \tag{1}$$

Solution 1. Recall that the cosine series for $|x|$ is

$$|x| = \frac{\pi}{2} - \sum_{n=0}^{\infty} \frac{4}{\pi(2n+1)^2} \cos(2n+1)x \quad \text{on} \quad [-\pi, \pi].$$

Since $|x|$ is continuous on $[-\pi, \pi]$, this series converges to the value of this function at $x = 0$, i.e., its value at 0 must be 0. This means

$$\frac{\pi}{2} = \sum_{n=0}^{\infty} \frac{4}{\pi(2n+1)^2}, \quad \text{so} \quad \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{8}.$$

Solution 2. Consider the function 1 on $[0, \pi]$ and its odd extension to $[-\pi, \pi]$. The resulting function f has a sine expansion, namely

$$f(x) = \sum_{n=1}^{\infty} \frac{4}{\pi(2n+1)} \sin(2n+1)x.$$

By Parseval's theorem,

$$1 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 \mathrm{dx} = \frac{1}{2} \sum_{n=0}^{\infty} \frac{16}{\pi^2(2n+1)^2},$$

which gives (1).

5. The Fourier series of an odd function consists of sine terms only. What additional condition on f implies that its sine coefficients with even indices vanish?

Solution. We must have $\int_0^\pi f(x)\sin(2jx)\,\mathrm{dx} = 0$ for all $j \in \mathbb{N}$. Since $\sin 2jx$ is antisymmetric around the point $x = \pi/2$, it will be sufficient that f be symmetric around the point $x = \pi/2$, i.e., that (the 2π-periodic extension of) f satisfy $f(x) = f(\pi - x)$ for all x. Then

$$\int_0^{\pi} f(x)\sin(2jx)\,\mathrm{dx} = \int_0^{\pi/2} f(x)\sin(2jx)\,\mathrm{dx} + \int_{\pi/2}^{\pi} f(x)\sin(2jx)\,\mathrm{dx}$$
$$= \int_0^{\pi/2} f(x)\sin(2jx)\,\mathrm{dx} - \int_0^{\pi/2} f(x)\sin(2jx)\,\mathrm{dx} = 0.$$

Examples of such functions are any $\sin(2n+1)x$ for $n \in \mathbb{N}$, the square wave from the 2nd solution to Problem 4, and the piecewise linear wave from class.