Solutions to homework # 4.

1. The sinc function is the Fourier transform of a compactly supported function. Identify it.

Solution. A very close variant of this question is actually answered in the notes. The function is

$$f(t) = \begin{cases} \sqrt{\frac{\pi}{2}} & t \in [-1,1] \\ 0 & \text{otherwise} \end{cases}$$

Indeed,

$$\hat{f}(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-1}^{0} \sqrt{\pi/2} e^{-\mathrm{i}\lambda x} \,\mathrm{dx} = \left. \frac{e^{-\mathrm{i}\lambda x}}{-2\mathrm{i}\lambda} \right|_{-1}^{1} = \frac{\sin\lambda}{\lambda} = \mathrm{sinc}\lambda.$$

2. Suppose f ogether with f', f'' and f''' are in $L^2(\mathbb{R})$. What rate of decay should we expect from $\widehat{f}(\lambda)$ as $\lambda \to \pm\infty$?

Solution. Using the interplay between the Fourier transform and differentiation, we see that f''' is in L^2 iff its Fourier transform is in L^2 iff

$$\int_{\mathbb{R}} |\lambda^3 \hat{f}(\lambda)|^2 \, d\lambda < \infty.$$

So, the product $\lambda^3 \widehat{f}(\lambda)$ must be square integrable. This does not quite translate into a pointwise estimate

$$|\widehat{f}(\lambda)| \leq \frac{\text{const}}{|\lambda|^3} \quad \text{as} \quad \lambda \to \pm\infty,$$

but it at a minimum rules out any possibility of the sort

$$|\widehat{f}(\lambda)| \geq \frac{\text{const}}{|\lambda|^{3-\varepsilon}} \quad \text{as} \quad \lambda \to \pm\infty, \quad \varepsilon \geq 0.$$

3. Are the filters $(L_1 f)(t) := \int_{t-1}^{t} f(x)\,\mathrm{dx}$, $(L_2 f)(t) := \int_0^1 x^2 f(t-x)\,\mathrm{dx}$ (a) linear (b) time-invariant (c) causal?

Solution. Both filters are indeed linear, time-invariant and causal, because both are convolution filters, i.e., $L_j f = g_j * f$, $j = 1, 2$, where functions g_j are supported on $\mathbb{R}_+$. The function g_2 is simply

$$g_2(x) := \begin{cases} x^2 & \text{for } x \in [0,1] \\ 0 & \text{otherwise,} \end{cases}$$

and g_1 is the indicator function of the interval $[0, 1]$, since

$$(L_1 f)(t) = \int_{-1}^{0} f(x+t)\,\mathrm{dx} = \int_0^1 f(t-x)\,\mathrm{dx} = \int_{-\infty}^{\infty} g_1(x) f(t-x)\,\mathrm{dx}.$$

4. Is the product $\sqrt{\Delta_a f \Delta_\alpha \widehat{f}}$ preserved under (a) dilation (b) modulation (c) translation of f?

Solution. Dilation of f by a factor r transforms $\Delta_a f$ into into $\frac{1}{r^2}\Delta_{ra} f$; it transforms $\Delta_\alpha \widehat{f}$ into $r^2 \Delta_{\alpha/r} \widehat{f}$. The resulting product is therefore

$$\sqrt{\Delta_{ra} f \Delta_{\alpha/r} \widehat{f}}$$

which generally differs from the original product, except in case $a = \alpha = 0$, when they coincide.

Modulation of f does not change the absolute value of f, hence does not change $\Delta_a f$, but it translates $\widehat{f}$, therefore turns $\Delta_\alpha \widehat{f}$ into $\Delta_{\alpha-\alpha_0} \widehat{f}$. The resulting product is therefore different from the original one.

Likewise, translating f shifts a by a certain valus a_0 to give $\Delta_{a-a_0} f$ but leads to modulation on the Fourier domain, hence does not change $\Delta_\alpha \widehat{f}$. The result is not the same as the original product.

Notice, however, that the form of the expression $\sqrt{\Delta_a f \Delta_\alpha \widehat{f}}$ stays the same under all three operations even though the parameters a and α may change.