Brief solutions to homework # 5.

1 (oversampling). Complete the following outline.

(a) Suppose f is a band-limited function with $\hat{f}(\lambda) = 0$ for $|\lambda| \geq \Omega$. Fix a number $a > 1$. Repeat the proof of the Shannon-Whittaker sampling theorem to show that

$$\hat{f}(\lambda) = \sum_{n=-\infty}^{\infty} c_{-n} e^{-in\pi\lambda/a\Omega} \quad \text{with} \quad c_{-n} := \frac{\pi}{\sqrt{2\pi} a\Omega} f\left(\frac{n\pi}{a\Omega}\right).$$

(b) Let $\hat{g}_a(\lambda)$ be the piecewise linear function whose graph is obtained by connecting the points $(-a\Omega, 0)$, $(-\Omega, 1)$, $(\Omega, 1)$, and $(a\Omega, 0)$. Show that

$$g_a(t) = \frac{\sqrt{2}(\cos(\Omega t) - \cos(a\Omega t))}{\sqrt{\pi}(a-1)\Omega t^2}$$

(c) Since $\hat{f}(\lambda) = 0$ for $|\lambda| \geq \Omega$, $\hat{f}(\lambda) = \hat{f}(\lambda)\hat{g}_a(\lambda)$. Use this fact, together with parts (a) and (b), to show that

$$f(t) = \sum_{n=-\infty}^{\infty} \frac{\pi}{\sqrt{2\pi} a\Omega} f\left(\frac{n\pi}{a\Omega}\right) g_a\left(t - \frac{n\pi}{a\Omega}\right).$$

Remark: Since $g_a(t)$ has the factor t^2 in the denominator, this expression for $f(t)$ converges faster than the expression for $f(t)$ given in the Shannon-Whittaker theorem. The disadvantage of the last formula is that the function is sampled on a denser grid. This is the tradeoff between the sample rate and the rate of convergence.

Solution is omitted, since the outline makes it very clear what to do.

2 (circulants and DFT). An $n \times n$ matrix A is called a circulant if $a(i,j)$ depends only on $(i-j) \bmod n$. For example, this matrix is a circulant:

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \\ 3 & 4 & 1 & 2 \\ 2 & 3 & 4 & 1 \end{bmatrix}.$$

(a) Look at the n-periodic sequence a where $a_l = a(l+1, 1)$, $l = 0, \ldots, n-1$. Write the entries of A in terms of the sequence a.

The (i,j)-entry $a(i,j)$ of A can be written as $a(i-j)$, i.e., it depends only on the difference between $i - j$ modulo n (recall that the sequence a is meant to be periodic with period n). Thus A is determined by its first column $a_0, a_1, \ldots, a_{n-1}$.

(b) Let X be a column vector with n components. Show that $Y = AX$ is equivalent to $y = a * x$ if x, y are periodic sequences for which $x_l = X_{l+1}$, $y_l = Y_{l+1}$ for $l = 0, \ldots, n-1$.

Just use the formula obtained in part (a): $y_i = \sum_j a(i-j) x_j$.

(c) Prove that the DFT diagonalizes all circulant matrices, i.e., that

$$\frac{1}{n}F_n^* A F_n \quad \text{is a diagonal matrix.}$$

What are the eigenvalues of A?

Using (b), we see that $\frac{1}{n}AF_n = F_n D$ where D is a diagonal matrix with diagonal entries $\sum_j a(i-j)\omega^{jk}$, for $k = 0, \ldots, n-1$. (Here $\omega = e^{2\pi \mathrm{i}/n}$ as usual.) These are precisely the eigenvalues of A.

3. Let L be the convolution operator associated with a sequence f, i.e., $Lx := f * x$. What is the adjoint of L (in the sense of l^2)?

Solution. Since $(Lx)(k) = \sum_j x(j), f(k-j)$, we have

$$\langle Lx, y \rangle = \sum_{j,k} x(j) f(k-j) \overline{y(k)} = \sum_j x(j) \overline{\sum_k \overline{f(k-j)} y(k)} = \langle x, L^* y \rangle,$$

we see that the adjoint L^* is also a convolution operator determined by the sequence $g(k) := \overline{f(-k)}$.