Solutions to homework # 6.

1. Let ϕ and ψ be the Haar scaling function and the Haar wavelet, respectively, and let the spaces V_j and W_j be defined as in the lectures. Let a function f be given by

$$f(x) = \begin{cases} -1 & 0 \le x < 1/4, \\ 4 & 1/4 \le x < 1/2, \\ 2 & 1/2 \le x < 3/4, \\ -3 & 3/4 \le x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

Express f in terms of the natural basis of V_2 and then decompose it into its V_0, W_0 and W_1 components.

Solution. We start at the given representation

$$f(x) = -\phi(2^2x) + 4\phi(2^2x-1) + 2\phi(2^2x-2) - 3\phi(2^2x-3)$$

and run the wavelet decomposition algorithm. Using the relations

$$\phi(2^{j+1}x-k) = \begin{cases} \frac{1}{2}\left(\phi(2^jx-\frac{k}{2}) + \psi(2^jx-\frac{k}{2})\right), & k \text{ even} \\ \frac{1}{2}\left(\phi(2^jx-\frac{k-1}{2}) - \psi(2^jx-\frac{k-1}{2})\right), & k \text{ odd,} \end{cases}$$

we get

$$f(x) = \frac{3}{2}\phi(2x) - \frac{1}{2}\phi(2x-1) - \frac{5}{2}\psi(2x) + \frac{5}{2}\psi(2x-1).$$

We keep the part $-\frac{5}{2}\psi(2x) + \frac{5}{2}\psi(2x-1)$, since it is in W_1, and further decompose the part $\frac{3}{2}\phi(2x) - \frac{1}{2}\phi(2x-1)$, which is in V_1. This produces

$$f(x) = \underbrace{\frac{1}{2}\phi(x)}_{\in V_0} + \underbrace{\psi(x)}_{\in W_0} \underbrace{- \frac{5}{2}\psi(2x) + \frac{5}{2}\psi(2x-1).}_{\in W_1}$$

2. What is the dimension of the spaces W_n and V_n restricted to the interval $[0,1]$?

Solution. Since the collection $\{2^{n/2}\phi(2^nx-k) : k \in \mathbb{Z}\}$ is an orthonormal basis for V_n, and the support of each function $\phi(2^nx-k)$ is the interval $[2^{-n}k, 2^{-n}(k+1)]$, the functions corresponding to $k = 0, \ldots, 2^n - 1$ form on orthonormal basis for $V_n|_{[0,1]}$. There are 2^n of them, so each space $V_n|_{[0,1]}$ has dimension 2^n. Now, $V_{n+1} = V_n \oplus W_n$, so

$$\dim W_n|_{[0,1]} = \dim V_{n+1}|_{[0,1]} - \dim V_n|_{[0,1]} = 2^{n+1} - 2^n = 2^n.$$

Answer. $\dim V_n|_{[0,1]} = \dim W_n|_{[0,1]} = 2^n$.

3. Reconstruct a function $f \in V_3$ given these coefficients in its Haar decomposition:

$$a^{[1]} = [3/2,\ -1], \quad b^{[1]} = [-1,\ -3/2], \quad b^{[2]} = [-3/2,\ -3/2,\ -1/2,\ -1/2].$$

The first entry in each list corresponds to $k = 0$. Sketch f.

Solution. We run the wavelet reconstruction algorithm using the rules

$$\begin{aligned} a^{[j]}2k &= a_k^{[j-1]} + b_k^{[j-1]}, \\ a^{[j]})2k+1 &= a_k^{[j-1]} - b_k^{[j-1]}. \end{aligned}$$

We need two iterations to arrive at level 3. We first use $a^{[1]}$ and $b^{[1]}$ to obtain

$$a^{[2]} = [1/2, 5/2, -5/2, 1/2].$$

Then we use $a^{[2]}$ and $b^{[2]}$ to get

$$a^{[3]} = [-1, 2, 1, 4, -3, -2, 0, 1].$$

Thus $f(x) = -\phi(2^3x) + 2\phi(2^3x - 1) + \phi(2^3x - 2) + 4\phi(2^3x - 3) - 3\phi(2^3x - 4) - 2\phi(2^3x - 5) + \phi(2^3x - 7)$. Below is a graph of this function

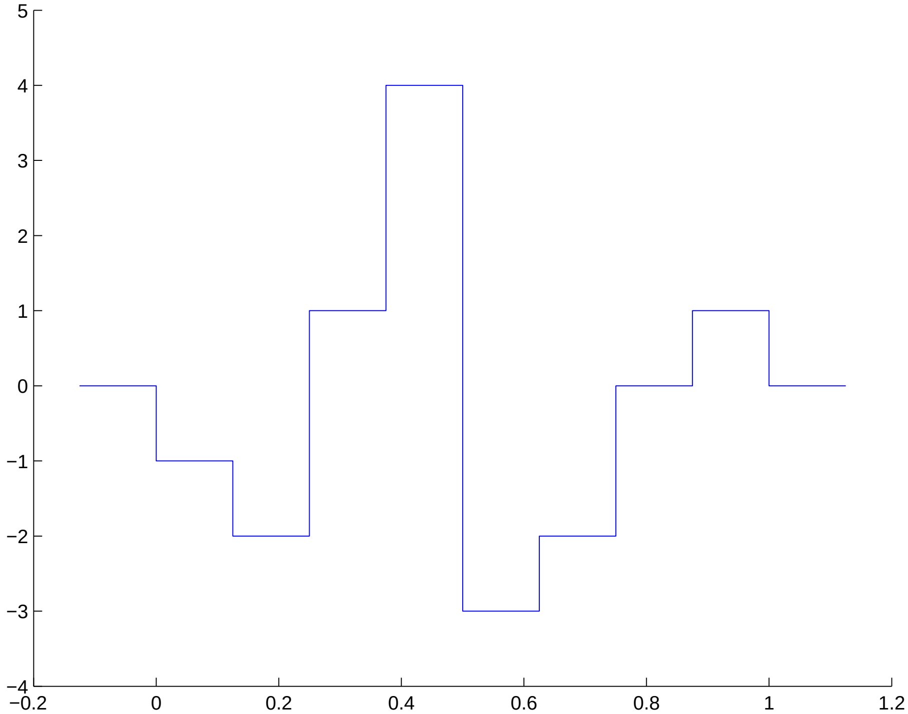

Figure 1. The recontructed function.

4. Consider the function $f(x) := x^2$ on $[0, 1]$. What dyadic level of discretization produces a piecewise-constant version f_d of f with relative error less than 0.1? The relative error is taken with respect to the $L^2[0, 1]$-norm, i.e., is equal to

$$\frac{\|f - f_d\|_{L^2[0,1]}}{\|f\|_{L^2[0,1]}}.$$

Solution. We should try to project f onto spaces V_j and take the first projection that gives the desired relative error. This is rather tedious and should be done in MATLAB or Maple.

First consider the orthogonal projection of f onto V_2. The coefficients $\sum a_{2,j}\phi(2^2x - j)$ are the integrals

$$a_{2,j} = 4\int_{\mathbb{R}} x^2\phi(2^2x - j)\mathrm{dx} \tag{1}$$

(notice that 4 occurs due to normalization $\{2\phi(2^2x - j)\}$). Using (1), we obtain

$$a_{2,0} = 1/48, \quad a_{2,1} = 7/48, \quad a_{2,2} = 19/48, \quad a_{2,3} = 37/48.$$

This yields relative error about 0.185, as can be verified numerically. So, this approximation is not good enough.

Projecting onto the next level V_3, using the analogous formula

$$a_{3,j} = 8\int_{\mathbb{R}} x^2\phi(2^3x - j)\mathrm{dx},$$

we get

$$\begin{aligned} a_{3,0} &= 1/192, & a_{3,1} &= 7/192, & a_{3,2} &= 19/192, & a_{3,3} &= 37/192,\\ a_{3,4} &= 61/192, & a_{3,5} &= 91/192, & a_{3,6} &= 127/192, & a_{3,7} &= 169/192, \end{aligned}$$

which gives relative error about $0.093 < 0.1$. Thus, we need a dyadic discretization at level 3 to produce an approximation to f with relative error smaller than 0.1.