

Solutions to homework # 7.

1. Let $P(z) = (1/2) \sum_k p_k z^k$ and define

$$Q(z) := -z \overline{P(-z)}.$$

Show that $Q(z) = (1/2) \sum_k (-1)^k \overline{p_{1-k}} z^k$. What is the relevance of this formula?

Solution. Recall that $|z| = 1$, hence $\bar{z} = 1/z$. This allows us to rewrite the formula for Q as follows:

$$\begin{aligned} Q(z) &= -z \overline{P(-z)} = -\frac{z}{2} \overline{\sum_k p_k (-z)^k} = -\frac{z}{2} \sum_k (-1)^k \overline{p_k} z^{-k} \\ &= \frac{1}{2} \sum_l (-1)^{2-l} \overline{p_{1-l}} z^l = \frac{1}{2} \sum_l (-1)^l \overline{p_{1-l}} z^l. \end{aligned}$$

This formula gives the mask for the wavelet ψ :

$$\widehat{\psi}(2\omega) = \left(\frac{1}{2} \sum_k (-1)^k \overline{p_{1-k}} e^{-ik\omega} \right) \widehat{\phi}(\omega).$$

2. Show that if a scaling function ϕ with orthonormal shifts has compact support, then there is only a finite number of nonzero coefficients in its scaling relation.

Solution. Since the shifts of ϕ are orthonormal, the coefficients p_k in the refinement equation

$$\phi(x) = \sum_k p_k \phi(2x - k)$$

are simply the inner products of ϕ with normalized shifts of $\phi(2\cdot)$:

$$p_k = 2 \int_{\mathbf{R}} \phi(x) \overline{\phi(2x - k)} dx.$$

Now, if ϕ is supported on $[-a, a]$, then $\phi(2\cdot - k)$ is supported on $[(k-a)/2, (k+a)/2]$. Hence the supports of ϕ and of $\phi(2\cdot - k)$ do not overlap for $|k| > 3a$. Hence $p_k = 0$ whenever $|k| > 3a$.

3. Show that if the p_k 's in the scaling relation are all zero for $k > N$ and $k < 0$ and the iterates ϕ_n in the Cascade algorithm converge to ϕ , then ϕ has compact support. Find the length of the support of ϕ in terms on N .

Solution. The cascade iterations use the formula

$$\phi_{n+1}(x) = \sum_{k=0}^N p_k \phi_n(2x - k). \tag{1}$$

Let us prove by induction that the support of ϕ_n lies within the interval $[0, (1+(2^n-1)N)/2^n]$. Indeed, this is true for $n = 0$, since the support of ϕ_0 , the Haar scaling function, is $[0, 1]$.

Now use (1) to justify the induction step. Since the leftmost function that contributes to the formation of ϕ_{n+1} is $\phi_n(2x)$, whose support begins at 0, we see that the support of ϕ_{n+1} cannot extend to the left of the origin. The rightmost function that contributes is $\phi_n(2x-N)$, whose support is by induction in the interval $[N/2, N/2+(1+(2^n-1)N)/2^{n+1}] = [N/2, (1+(2^{n+1}-1)N)/2^{n+1}]$, so the support of ϕ_{n+1} does not extend to the right of the point $(1+(2^{n+1}-1)N)/2^{n+1}$. This proves our claim.

Taking the limit as $n \rightarrow \infty$, we see that the support of the resulting refinable function ϕ must be in the interval $[0, N]$.

4. Using MATLAB, run the Cascade algorithm for the mask $P(z) = (1/2) \sum_{k=0}^3 p_k z^k$ where

$$p_0 = \frac{1 + \sqrt{3}}{4}, \quad p_1 = \frac{3 + \sqrt{3}}{4}, \quad p_2 = \frac{3 - \sqrt{3}}{4}, \quad p_3 = \frac{1 - \sqrt{3}}{4}.$$

Graph the resulting function. How smooth is it?

Solution (contributed by David Bolin).

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p0 = (1+sqrt(3))/4;
p1 = (3+sqrt(3))/4;
p2 = (3-sqrt(3))/4;
p3 = (1-sqrt(3))/4;
p = [p0, p1, p2, p3]
figure(1)
itr = 20;
phi= p;
for i=1:itr
    phitemp = [];
    for j = 1:4
        phitemp(j,:)= [zeros(1,2^(i)*(j-1)), p(j)*phi, zeros(1,2^(i)*(4-j))];
    end
    phi = phitemp(1,:)+ phitemp(2,:)+ phitemp(3,:)+ phitemp(4,:);
end
X = linspace(0,(1+(2^(itr)-1)*3)/(2^(itr)),length(phi));
plot(X,phi)
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The way the program calculates ϕ_{n+1} can be basically seen as upsampling of ϕ_n followed by convolution by the vector p . The convergence rate is fast, after seven iterations it is getting hard to see any difference in the figures. The figure below is given by 10 iterations, we see that the function is continuous but not differentiable.

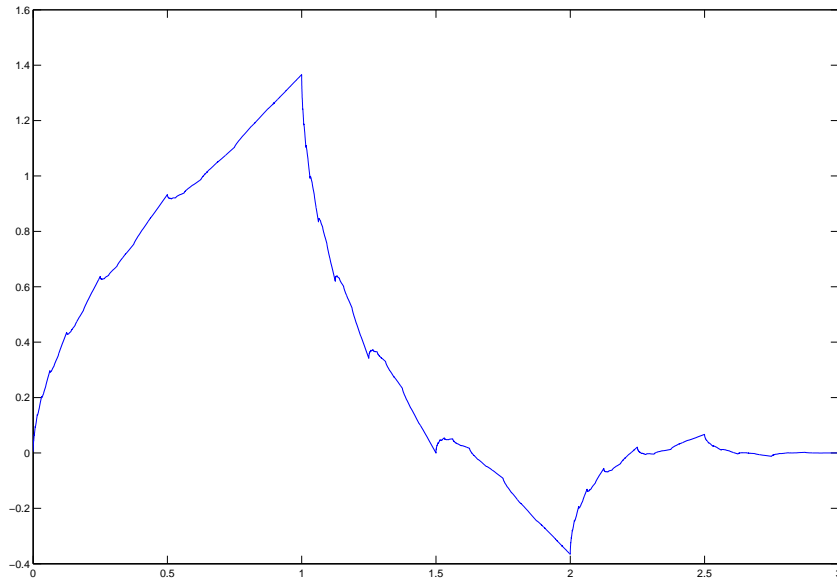


Figure 1. Daubechies-2 refinable function.