Solutions to homework # 9.

1. Calculate explicitly (by hand or using symbolic math software) the polynomial pieces that make up the cardinal spline $B_3 = B_1 * B_1 * B_1$.

Solution. $B_3 = B_2 * B_1$, i.e., B_3 is obtained by convolving the hat function B_2 with the box function B_1. Using the formula

$$B_2(x) = \begin{cases} x, & x \in [0,1], \\ 2-x, & x \in [1,2], \end{cases}$$

we obtain

$$B_3(x) = \begin{cases} \int_0^x y\mathrm{d}y, & x \in [0,1], \\ \int_{x-1}^1 y\mathrm{d}y + \int_1^x (2-y)\mathrm{d}y, & x \in [1,2], \\ \int_{x-1}^x (2-y)\mathrm{d}y, & x \in [2,3], \\ 0 & \text{otherwise.} \end{cases}$$

This gives, after a bit of calculus, the formula

$$B_2(x) = \begin{cases} x^2/2, & x \in [0,1], \\ (6x - 2x^2 - 3)/2, & x \in [1,2], \\ (x-3)^2/2, & x \in [2,3], \\ 0 & \text{otherwise.} \end{cases}$$

2. Compute all eigenvectors and the spectral radius of the transfer operator for the 3rd Daubechies scaling function coming from the expansion of

$$(\cos^2(\omega/2) + \sin^2(\omega/2))^5.$$

How many derivatives of this function are in L^2?

Solution (provided by David Bolin). Using the method described in Amos Ron's notes, we see that the original transfer map can be restricted to the linear span of

$$\{e^{\mathrm{i}k\omega} : k = -5, \ldots, 5\},$$

since

$$\mathcal{H}(\omega) = \cos^6(\omega/2)\left(\cos^4(\omega/2) + 5\cos^2(\omega/2)\sin^2(\omega/2) + 10\sin^4(\omega/2)\right).$$

Its matrix then has the form

$$T = \frac{1}{256}\begin{bmatrix} 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -25 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 150 & 0 & -25 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 150 & 256 & 150 & 0 & -25 & 0 & 3 & 0 & 0 & 0 & 0 \\ -25 & 0 & 150 & 256 & 150 & 0 & -25 & 0 & 3 & 0 & 0 \\ 3 & 0 & -25 & 0 & 150 & 256 & 150 & 0 & -25 & 0 & 3 \\ 0 & 0 & 3 & 0 & -25 & 0 & 150 & 256 & 150 & 0 & -25 \\ 0 & 0 & 0 & 0 & 3 & 0 & -25 & 0 & 150 & 256 & 150 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & -25 & 0 & 150 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & -25 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 \end{bmatrix}.$$

This matrix can be obtained using MATLAB symbolic toolbox (see below).

Factoring $\cos^6(\omega/2)$, we obtain the new transfer map corresponding to the function

$$\mathcal{H}_1(\omega) = \cos^4(\omega/2) + 5\cos^2(\omega/2)\sin^2(\omega/2) + 10\sin^4(\omega/2).$$

It acts on the space spanned by

$$\{e^{ik\omega} : k = -2, \dots, 2\}$$

and its matrix is

$$T_1 = \begin{bmatrix} 3/4 & 0 & 0 & 0 & 0 \\ 19/2 & -9/2 & 3/4 & 0 & 0 \\ 3/4 & -9/2 & 19/2 & -9/2 & 3/4 \\ 0 & 0 & 3/4 & -9/2 & 19/2 \\ 0 & 0 & 0 & 0 & 3/4 \end{bmatrix}.$$

Here is the MATLAB script that was used to produce these results.

```
%the script last calculates the transfer operators T and T1
%and their eigenvalues and eigenvectors.

om = sym('om');
e1 = 1/2*exp(1/4*i*om)+1/2*exp(-1/4*i*om);
e2 = (1/(2*i))*exp(1/4*i*om)-(1/(2*i))*exp(-1/4*i*om);
H1 = ((e1)^6)*((e1)^4+5*(e1^2)*(e2^2)+10*(e2^4));
H2 = ((e2)^6)*((e2)^4+5*(e2^2)*(e1^2)+10*(e1^4));

p0 = simple(H1*exp(-(i*5/2)*om)-H2*exp(-(i*5/2)*om));
p1 = simple(H1*exp(-(i*4/2)*om)+H2*exp(-(i*4/2)*om));
p2 = simple(H1*exp(-(i*3/2)*om)-H2*exp(-(i*3/2)*om));
p3 = simple(H1*exp(-(i*2/2)*om)+H2*exp(-(i*2/2)*om));
p4 = simple(H1*exp(-(i*1/2)*om)-H2*exp(-(i*1/2)*om));
p5 = simple(H1+H2);
p6 = simple(H1*exp((i*1/2)*om)-H2*exp((i*1/2)*om));
```

```
p7 = simple(H1*exp((i*2/2)*om)+H2*exp((i*2/2)*om));
p8 = simple(H1*exp((i*3/2)*om)-H2*exp((i*3/2)*om));
p9 = simple(H1*exp((i*4/2)*om)+H2*exp((i*4/2)*om));
p10 = simple(H1*exp((i*5/2)*om)-H2*exp((i*5/2)*om));

%By looking at the coefficients for p_j we get the following
%matrix T:

T0 = [3/256; -25/256; 75/128; 75/128; -25/256; 3/256; 0;0;0;0;0;];
T1 = [0 0 0 1 0 0 0 0 0 0 0]';
T2 = [0; T0(1:6);0;0;0;0];
T3 = [0 0 0 0 1 0 0 0 0 0 0]';
T4 = [0; 0; T0(1:6);0;0;0];
T5 = [0 0 0 0 0 1 0 0 0 0 0]';
T6 = [0;0;0; T0(1:6);0;0];
T7 = [0 0 0 0 0 0 1 0 0 0 0]';
T8 = [0;0;0;0; T0(1:6);0];
T9 = [0 0 0 0 0 0 0 1 0 0 0]';
T10 = [0;0;0;0;0; T0(1:6)];
%
T = [T0, T1, T2, T3, T4, T5, T6, T7, T8, T9, T10];

%we get the following eigenvectors and eigenvalues:

[V,D] = eig(T)

%We now calculate T1

H11 = ((e1)^4+5*(e1^2)*(e2^2)+10*(e2^4));
H12 = ((e2)^4+5*(e2^2)*(e1^2)+10*(e1^4));


q0 = simple(H11*exp(-(i*2/2)*om)+H12*exp(-(i*2/2)*om));
q1 = simple(H11*exp(-(i*1/2)*om)-H12*exp(-(i*1/2)*om));
q2 = simple(H11+H12);
q3 = simple(H11*exp((i*1/2)*om)-H12*exp((i*1/2)*om));
q4 = simple(H11*exp((i*2/2)*om)+H12*exp((i*2/2)*om));

%looking at the coefficients in q_j we get the matrix for T1:

T1 = [3/4 0 0 0 0; 19/2 -9/2  3/4 0 0; 3/4 -9/2 19/2 -9/2 3/4;
0 0 3/4 -9/2 19/2; 0 0 0 0 3/4];

%We get the following eigenvectors and eigenvalues.
```

```
[V1,D1] = eig(T1)
```

And here is the output of this script.

```
last

V =

  Columns 1 through 7

         0         0         0         0         0         0         0
         0   -0.0003    0.0007    0.0341   -0.0297   -0.0196    0.0069
         0   -0.0136    0.0158   -0.2048    0.1582   -0.0524    0.0367
         0    0.1352   -0.1212    0.4780   -0.3559    0.4060   -0.2354
         0   -0.6939    0.4688   -0.4780    0.4746   -0.5762    0.5015
    1.0000   -0.0000   -0.7284    0.0000   -0.4943    0.0000   -0.6192
         0    0.6939    0.4688    0.4780    0.4746    0.5762    0.5015
         0   -0.1352   -0.1212   -0.4780   -0.3559   -0.4060   -0.2354
         0    0.0136    0.0158    0.2048    0.1582    0.0524    0.0367
         0    0.0003    0.0007   -0.0341   -0.0297    0.0196    0.0069
         0         0         0         0         0         0         0

  Columns 8 through 11

         0         0    0.0736         0
   -0.0020    0.0030   -0.2982    0.0104
   -0.0242    0.0320    0.3155    0.0104
    0.1858   -0.3584    0.2551   -0.1376
   -0.4928    0.6087   -0.7373    0.0972
    0.6664    0.0000    0.4108    0.4108
   -0.4928   -0.6087    0.0972   -0.7373
    0.1858    0.3584   -0.1376    0.2551
   -0.0242   -0.0320    0.0104    0.3155
   -0.0020   -0.0030    0.0104   -0.2982
         0         0         0    0.0736


D =

  Columns 1 through 7

    1.0000         0         0         0         0         0         0
         0    0.5000         0         0         0         0         0
         0         0    0.2500         0         0         0         0
         0         0         0   -0.0703         0         0         0
```

```
         0         0         0         0   -0.0625         0         0
         0         0         0         0         0    0.0313         0
         0         0         0         0         0         0    0.0625
         0         0         0         0         0         0         0
         0         0         0         0         0         0         0
         0         0         0         0         0         0         0
         0         0         0         0         0         0         0

  Columns 8 through 11

         0         0         0         0
         0         0         0         0
         0         0         0         0
         0         0         0         0
         0         0         0         0
         0         0         0         0
         0         0         0         0
    0.1406         0         0         0
         0    0.1250         0         0
         0         0    0.0117         0
         0         0         0    0.0117


V1 =

         0         0         0    0.4150         0
   -0.0554    0.6396    0.7071    0.8096    0.0587
   -0.9969    0.4264    0.0000    0.4110    0.4110
   -0.0554    0.6396   -0.7071    0.0587    0.8096
         0         0         0         0    0.4150


D1 =

    9.0000         0         0         0         0
         0   -4.0000         0         0         0
         0         0   -4.5000         0         0
         0         0         0    0.7500         0
         0         0         0         0    0.7500

diary off
```

We see that the spectral radius of the second transfer map is 9, hence the smoothness of the Daubechies-3 refinable function is

$$n - \frac{\log_2 \varrho(T_1)}{2} = 3 - \frac{\log_2 9}{2} \approx 1.415.$$

Thus the refinable function itself and its first derivative are in $L^2(\mathrm{I\!R})$ but higher order derivatives are not.