I. The matrix is symmetric, so is guaranteed to be diagonalizable. Hence our problem is equivalent to finding its eigenvalues and eigenvectors. Solving the characteristic equation, we get

$$\det \begin{bmatrix} 1-\lambda & -1 \\ -1 & 1-\lambda \end{bmatrix} = (1-\lambda)^2 - 1 = \lambda^2 - 2\lambda = \lambda(\lambda-2),$$

we get two eigenvalues, 0 and 2. An eigenvector corresponding to eigenvalue 0 can be found from the condition $x - y = 0$, so we can take the vector $[1\ \ 1]'$. An eigenvector orresponding to eigenvalue 2 can be found from the condition $x - y = 2x$, so we take $[1\ -1]'$.

Here is the obtained similarity transformation that makes A into a diagonal matrix:

$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}.$$

II. The integrand is even, hence the integral is equal to half of the integral over the whole real line:

$$\int_0^\infty \frac{x^2}{\cosh ax}\,\mathrm{d}x = \frac{1}{2}\int_{-\infty}^\infty \frac{x^2}{\cosh ax}\,\mathrm{d}x.$$

Consider the rectangular contour $[-R, R]$, $[R, R+\pi\mathrm{i}/a]$, $[R+\pi\mathrm{i}/a, -R+\pi\mathrm{i}/a]$, $[-R+\pi\mathrm{i}/a, -R]$. The contribution from the "sides" tends to zero as $R \to \infty$, and the integral over the upper segment (taking into account orientation) is equal to

$$\int_{-R}^R \frac{(x+\pi\mathrm{i}/a)^2}{\cosh ax}\,\mathrm{d}x = \int_{-R}^R \frac{x^2 + 2\pi\mathrm{i}x/a - \pi^2/a^2}{\cosh ax}\,\mathrm{d}x = \int_{-R}^R \frac{x^2 - \pi^2/a^2}{\cosh ax}\,\mathrm{d}x,$$

since the second part gives rise to an odd function, whose integral over a symmetric interval is zero. The remaining integral was evaluated in Example 2.22. Its limit as $R \to \infty$ is $-\pi^3/a^3$. So,

$$2\int_{-\infty}^\infty \frac{x^2}{\cosh ax}\,\mathrm{d}x - \frac{\pi^3}{a^3} = 2\pi\mathrm{i}\,\mathrm{Res}\left(\frac{z^2}{\cosh az}\right)\bigg|_{z=\pi\mathrm{i}/2a} = 2\pi\mathrm{i}\,\frac{z^2}{a\sinh az}\bigg|_{z=\pi\mathrm{i}/2a} = \frac{-2\pi^3\mathrm{i}}{4a^3\mathrm{i}} = \frac{-\pi^3}{2a^3}.$$

Hence the original integral is equal to

$$\frac{1}{4}\left(\frac{\pi^3}{a^3} - \frac{\pi^3}{2a^3}\right) = \frac{\pi^3}{8a^3}.$$

III. The characteristic equation of the corresponding homogeneous ODE is

$$\lambda^2 - 2\lambda + 1 = (\lambda-1)^2 = 0.$$

It has a double root $\lambda = 1$, so the general solution to the homogeneous equation is $(A+Bx)e^x$. A particular solution to the given inhomogeneous ODE is $\frac{1}{2}\cos x$. Hence the general solution to the given equation is

$$(A+Bx)e^x - \frac{1}{2}\cos x.$$

To match the initial conditions, we must have

$$A - \frac{1}{2} = 1, \qquad A + B = 0,$$

so $A = 1/2$, $B = -1/2$.

IV. By the assumption of the problem, f and g are periodic with period 1, hence their Fourier series on the interval $[0,1]$ give correct representations of the two functions on the set of all reals. Let

$$f(x) = \sum_{k=-\infty}^{\infty} c_k e^{2\pi \mathrm{i} kx}, \qquad g(x) = \sum_{l=-\infty}^{\infty} d_l e^{2\pi \mathrm{i} lx}.$$

Then $g(nx) = \sum_{l=-\infty}^{\infty} d_l e^{2\pi \mathrm{i} nlx}$, and the integral in the left hand side is equal to

$$\sum_{l,k=-\infty}^{\infty} \int_0^1 c_k d_l e^{2\pi \mathrm{i}(k+nl)x}\,\mathrm{d}x = \sum_{l=-\infty}^{\infty} c_{-nl} d_l.$$

As $n \to \infty$, this infinite sum tends to $c_0 d_0$, which is nothing but $\int_0^1 f(x)\,\mathrm{d}x \int_0^1 g(x)\,\mathrm{d}x$.

V. Let f be any test function. By the rule of differentiation for distributions,

$$\begin{aligned}
\int_{-\infty}^{\infty} (e^{-x} + 2x)\delta''(x) f(x)\,\mathrm{d}x &= -\int_{-\infty}^{\infty} \delta'(x) \left((e^{-x} + 2x) f(x)\right)'\,\mathrm{d}x \\
&= -\int_{-\infty}^{\infty} \delta'(x) \left((-e^{-x} + 2) f(x) + (e^{-x} + 2x) f'(x)\right)\,\mathrm{d}x \\
&= \int_{-\infty}^{\infty} \delta(x) \left((-e^{-x} + 2) f(x) + (e^{-x} + 2x) f'(x)\right)'\,\mathrm{d}x \\
&= \int_{-\infty}^{\infty} \delta(x) \left(e^{-x} f(x) + 2(-e^{-x} + 2) f'(x) + (e^{-x} + 2x) f''(x)\right)\,\mathrm{d}x \\
&= f(0) + 2f'(0) + f''(0) = \int_{-\infty}^{\infty} \left(\delta(x) - 2\delta'(x) + \delta''(x)\right) f(x)\,\mathrm{d}x.
\end{aligned}$$

Hence $(e^{-x} + 2x)\delta''(x) = \delta(x) - 2\delta'(x) + \delta''(x)$.

VI. **(a)** From the table of basic Laplace transforms, $\sin 2t$ maps to $2/(s^2+4)$, $\cosh t$ to $1/2(1/(s-1) + 1/(s+1))$, and so by the differentiation rules, the transform of the given function is equal to

$$-\left(\frac{2}{s^2+4}\right)' + \frac{1}{2}\left(\frac{1}{s-1} + \frac{1}{s+1}\right)'' = \frac{4s}{(s^2+4)^2} + \frac{1}{(s-1)^3} + \frac{1}{(s+1)^3}.$$

(b) From the table of basic transforms, we see that $1 - \cos t$ gets transformed into

$$\frac{1}{s} - \frac{s}{s^2+1},$$

and the integration rules gives the transform of $(1 - \cos t)/t$ as

$$\int_s^{\infty} \left(\frac{1}{\sigma} - \frac{\sigma}{\sigma^2+1}\right) \mathrm{d}\sigma = \left(\ln \sigma - \frac{1}{2}\ln(\sigma^2+1)\right)\Big|_s^{\infty} = \ln \frac{\sigma}{\sqrt{\sigma^2+1}}\Big|_s^{\infty} = \ln \frac{\sqrt{s^2+1}}{s}.$$

VII. Let us denote the hat function by g. Notice that g is the convolution of the function

$$f(x) = \begin{cases} 1, & 0 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

with itself, i.e., $g = f * f$, so by the convolution rule,

$$\widehat{g} = (f * f)^{\wedge} = \sqrt{2\pi} \cdot \widehat{f} \cdot \widehat{f} = \sqrt{2\pi} \left(\widehat{f}\right)^2 .$$

The Fourier transform $\widehat{f}$ of f is easy to find:

$$\widehat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_0^1 e^{-\mathrm{i}\xi t}\, \mathrm{d}t = \frac{1}{\sqrt{2\pi}} \frac{1 - e^{-\mathrm{i}\xi}}{\mathrm{i}\xi},$$

so

$$\widehat{g}(\xi) = \frac{1}{\sqrt{2\pi}} \left(\frac{1 - e^{-\mathrm{i}\xi}}{\mathrm{i}\xi} \right)^2 .$$