Solutions to the final exam.

1. (5 pts.) Find an orthogonal similarity transformation that reduces the matrix

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

to a diagonal form.

Solution. A is a symmetric matric, so is known to be unitarily (orthogonally) similar to a diagonal matrix. The characteristic polynomial of A is $(\lambda - 1)(\lambda - 3)$, so A has two eigenvalues: 1 and 3. An eigenvector corresponding to eigenvalue 1 is $[1 \ \ -1]'$ and one corresponding to 3 is $[1\ 1]'$. They are already orthogonal, so it remains to normalize them by dividing by their length, which is in both cases equal to $\sqrt{2}$, to manufacture an orthogonal matrix. The final transformation is

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}.$$

2. (8 pts.) Find all zeros and all singularities of the function

$$\frac{\tan z}{z},$$

classify the singularities, indicate the order of each zero and each pole.

Solution. Look at $\tan z = \sin z / \cos z$ first. The numerator $\sin z$ vanishes at πk, $k \in \mathbb{Z}$, while the denominator vanishes at $\pi/2 + \pi k$, $k \in \mathbb{Z}$. All these points are simple zeros of the numerator or denominator, respectively. The simple zero of the numerator at $z = 0$ matches the simple zero of the function z, hence the given function has

- a removable singularity at $z = 0$;
- simple zeros at the points $z = \pi k$, $k \in \mathbb{Z} \backslash \{0\}$;
- simple poles at the points $z = \pi/2 + \pi k$, $k \in \mathbb{Z}$.

3. (10 pts.) Find a two-parameter family of solutions to the equation

$$y'' - xy = x^2$$

as power series in x.

Solution. First note that a particular solution to the equation is $-x$, so we only need two linearly independent solutions to the corresponding homogeneous equation

$$y'' - xy = 0. \tag{1}$$

Writing

$$y(x) = \sum_{n=0}^{\infty} a_n x^n,$$

integrating twice term by term, and substituting into (1), we get

$$2a_2 + \sum_{n=1} ((n+2)(n+1)a_{n+2} - a_{n-1})x^n = 0.$$

It follows that $a_2 = 0$ and for all $n \geq 1$, the recurrence

$$a_{n+2} = \frac{a_{n-1}}{(n+1)(n+2)} \tag{2}$$

holds. This shows, in particular, that $a_{2+3n} = 0$ for all $n \in \mathbb{Z}_+$. The constants a_0 and a_1 can be chosen arbitrarily, and the remaining ones will be all determined by those two from the recurrence (2). Precisely, if we introduce a notation for the "modified factorial"

$$k!!! := k(k-3)(k-6)\cdots,$$

with the understanding that that factorial is one if $k \leq 0$, then $a_n = \frac{(n-2)!!!}{n!} a_0$ for n divisible by 3 and $a_n = \frac{(n-2)!!!}{n!} a_1$ for n that is 1 modulo 3, so the answer to the problem is

$$-x + a_0 \sum_{n=0}^{\infty} \frac{(3n-2)!!!}{3n!} x^{3n} + a_1 \sum_{n=0}^{\infty} \frac{(3n-1)!!!}{(3n+1)!} x^{3n+1}.$$

4. (10 pts.) Let f be a twice differentiable real-valued function on $[0, 2\pi]$, with $f(2\pi) = f(0)$ and $\int_0^{2\pi} f(x)\,\mathrm{d}x = 0$. Show that

$$\int_0^{2\pi} (f(x))^2\,\mathrm{d}x \leq \int_0^{2\pi} (f'(x))^2\,\mathrm{d}x.$$

Solution. By the assumption of the problem, the function f Fourier series on the interval $[0, 2\pi]$ that may be differentiated term by term to produce a series for f'. Moreover, the zeroth coefficient in the Fourier series is zero, since $\int_0^{2\pi} f(x)\,\mathrm{d}x = 0$. So,

$$f(x) = \sum_{n \in \mathbb{Z},\ n \neq 0} c_n e^{\mathrm{i}nx}.$$

Differentiating term by term, we get

$$f'(x) = \sum_{n \in \mathbb{Z},\ n \neq 0} \mathrm{i}nc_n e^{\mathrm{i}nx}.$$

Now, by the Parseval formula, the left-hand side is equal to

$$2\pi \sum_{n \in \mathbb{Z},\ n \neq 0} |c_n|^2,$$

and the right-hand side is equal to

$$2\pi \sum_{n\in\mathbb{Z},\, n\neq 0} |inc_n|^2 = 2\pi \sum_{n\in\mathbb{Z},\, n\neq 0} n^2|c_n|^2,$$

which is indeed larger than the left-hand side.

5. (10 pts.) Find the Laplace transform of

$$f(t) = \int_0^t \left(e^{-2\tau}\sin^2\tau + \frac{\sin\tau}{\tau}\right) \mathrm{d}\tau.$$

Solution. First recall that we only need to find the Laplace transform (call it $F(s)$) of the integrand, since the transform of the integral is then equal to $F(s)/s$. Also recall that

$$\sin^2 t = \frac{1}{2}(1-\cos 2t),$$

so the Laplace transfor of $\sin^2 t$ is equal to

$$\frac{1}{2}\left(\frac{1}{s} - \frac{s}{s^2+4}\right).$$

Multiplication by e^{-2t} shifts the transform by 2 so that the transform of $e^{-2t}\sin^2 t$ is equal to

$$\frac{1}{2}\left(\frac{1}{s+2} - \frac{s+2}{(s+2)^2+4}\right).$$

Now, the transform of $\sin t$ is

$$\frac{1}{s^2+1,}$$

hence the transform of $\frac{\sin t}{t}$ is

$$\int_s^\infty \frac{\mathrm{d}\sigma}{\sigma^2+1} = \arctan\sigma\Big|_s^\infty = \frac{\pi}{2} - \arctan s.$$

Thus,

$$F(s) = \frac{1}{2(s+2)} - \frac{s+2}{2((s+2)^2+4)} + \frac{\pi}{2} - \arctan s.$$

and the answer is

$$\frac{F(s)}{s} = \frac{1}{2s(s+2)} - \frac{s+2}{2s((s+2)^2+4)} + \frac{\pi}{2s} - \frac{\arctan s}{s}.$$

6. (7 pts.) Express the distribution $(e^{2x}+x^2)\delta''(x)$ as a constant-coefficient linear combination of $\delta(x)$ and its derivatives.

Solution. Integrating against a test function f, we get

$$\begin{aligned}\int_{-\infty}^{\infty}(e^{2x}+x^2)\delta''(x)f(x)\,\mathrm{d}x &= \int_{-\infty}^{\infty}\delta(x)\left((e^{2x}+x^2)f(x)\right)''\,\mathrm{d}x\\ &= \int_{-\infty}^{\infty}\delta(x)\left((4e^{2x}+2)f(x)+2(2e^{2x}+2x)f'(x)+(e^{2x}+x^2)f''(x)\right)\,\mathrm{d}x\\ &= 6f(0)+4f'(0)+f''(0).\end{aligned}$$

So, $(e^{2x}+x^2)\delta''(x) = 6\delta(x) - 4\delta'(x) + \delta''(x)$.

7. (10 pts.) Find the Fourier transform of

$$f(x) = \frac{1}{\cosh ax}, \qquad a > 0.$$

Solution. We need to evaluate

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{e^{-\mathrm{i}\xi x}\,\mathrm{d}x}{\cosh ax}.$$

Consider the rectangular contour C_R with base $[-R, R]$ and top $[-R+\pi\mathrm{i}/a.R+\pi\mathrm{i}/a]$. The shift $x \mapsto x+\pi\mathrm{i}/a$ reverses the sign of the denominator $\cosh ax$, the numerator gets multiplied by $e^{\pi\xi/a}$, and the direction of integration is from right to left. The integrals along the sides of the contour tend to zero, so all together

$$(1+e^{\pi\xi/a})\int_{-\infty}^{\infty}\frac{e^{-\mathrm{i}\xi x}}{\cosh ax}\,\mathrm{d}x = \lim_{R\to\infty}\oint_{C_R}\frac{e^{-\mathrm{i}\xi z}}{\cosh az}\,\mathrm{d}z.$$

By the Residue theorem, the contour is equal to $2\pi\mathrm{i}$ times the residue of the integrand at the point $z = \pi\mathrm{i}/2a$, which is its simple pole. This residue is therefore

$$\operatorname{Res}\frac{e^{-\mathrm{i}\xi z}}{\cosh az}\bigg|_{z=\pi\mathrm{i}/2a} = \frac{e^{-\mathrm{i}\xi z}}{a\sinh az}\bigg|_{z=\pi\mathrm{i}/2a} = \frac{e^{\pi\xi/2a}}{\mathrm{i}a}.$$

So, the sought-for Fourier transform is

$$\frac{1}{\sqrt{2\pi}}\cdot\frac{2\pi\mathrm{i}}{1+e^{\pi\xi/a}}\cdot\frac{e^{\pi\xi/2a}}{\mathrm{i}a} = \frac{\sqrt{\pi}}{\sqrt{2}a\cosh(\pi\xi/2a)}.$$