**35.** First note $\vec{\mathbf{u}}_1 + \vec{\mathbf{u}}_2 = \vec{\mathbf{u}}_3$, i.e., the vectors $\vec{\mathbf{u}}_1$, $\vec{\mathbf{u}}_2$, $\vec{\mathbf{u}}_3$ are linearly dependent, i.e., they lie in the same plane. The vectors $\vec{\mathbf{u}}_1$ and $\vec{\mathbf{u}}_2$, on the other hand, are not not proportional, hence linearly independent. So, it is enough to determine whether $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ can be expressed as linear combinations of $\vec{\mathbf{u}}_1$ and $\vec{\mathbf{u}}_2$ *only.* Since the second component of $\vec{\mathbf{u}}_2$ is zero, the coefficient of $\vec{\mathbf{u}}_1$ must match the second component of a vector we want to write as a linear combination of $\vec{\mathbf{u}}_1$ and $\vec{\mathbf{u}}_2$. So, for $\vec{\mathbf{a}}$ we must take $2\vec{\mathbf{u}}_1$, and $\vec{\mathbf{a}} - 2\vec{\mathbf{u}}_1 = -\vec{\mathbf{u}}_2$, so $\vec{\mathbf{a}} = 2\vec{\mathbf{u}}_1 - \vec{\mathbf{u}}_2$. Geometrically, this means that $\vec{\mathbf{a}}$ lies in the same plane as $\vec{\mathbf{u}}_1$, $\vec{\mathbf{u}}_2$, and $\vec{\mathbf{u}}_3$.

On the other hand, to express $\vec{\mathbf{b}}$, we must take $3\vec{\mathbf{u}}_1$, but $\vec{\mathbf{a}} - 3\vec{\mathbf{u}}_1 = (-2, 0, -1)$ is not a multiple of $\vec{\mathbf{u}}_2$, so $\vec{\mathbf{b}}$ cannot be expressed as a combination of $\vec{\mathbf{u}}_1$, $\vec{\mathbf{u}}_2$, and $\vec{\mathbf{u}}_3$. Geometrically, this means that $\vec{\mathbf{b}}$ does not lie in the plane defined by these vectors.

**40.** We know that $\mathrm{tr}(AB) = \mathrm{tr}(BA)$ (see p. 44 of the textbook). Hence

$$\mathrm{tr}(CAC^{-1}) = \mathrm{tr}((CA)C^{-1}) = \mathrm{tr}(C^{-1}(CA)) = \mathrm{tr}((C^{-1}C)A) = \mathrm{tr}(IA) = \mathrm{tr}(A).$$

Here $I$ denotes the identity matrix and the associativity of matrix products is used. Note that the product of matrices is generally not commutative, so we cannot rearrange the factors $C$, $A$, and $C^{-1}$ at will.

**48.** A matrix $A$ is real symmetric if and only if it equals its transpose: $A = A^T$. A matrix $C$ is orthogonal if and only if $C^{-1} = C^T$. Also note that $(A^T)^T$ and $(ABC)^T = C^T B^T A^T$ for any matrices $A$, $B$, $C$. Therefore,

$$(CAC^{-1})^T = (C^{-1})^T A^T C^T = (C^T)^T AC^{-1} = CAC^{-1}.$$

So, the matrix $CAC^{-1}$ is symmetric.

**49.** If $A$, $B$ are diagonal $n \times n$-matrices, then

$$A = \begin{bmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n \end{bmatrix}, \; B = \begin{bmatrix} b_1 & 0 & \cdots & 0 \\ 0 & b_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b_n \end{bmatrix}, \; \text{so } AB = \begin{bmatrix} a_1b_1 & 0 & \cdots & 0 \\ 0 & a_2b_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_nb_n \end{bmatrix} = BA.$$

**50.** First recall that the inner product of two column-vectors $x$ and $y$ is simply the result of the matrix operation $x^T y$. If the change of basis $C$ is applied, then the vectors $x$ and $y$ become $Cx$ and $Cy$. Recall that $C$ is assumed to be orthogonal, i.e., $CC^T = I$. Therefore, for all vectors $x$ and $y$, we have

$$(Cx)^T (Cy) = x^T C^T Cy = x^T (C^T C)y = x^T Iy = x^T y.$$

Thus, an orthogonal change of basis preserves the inner product.

**53.** Choose the point $P$ where the first rod is attached as a reference point, and let the $x$-coordinate point horizontally to the right of $P$, and the $y$-coordinate vertically down. Then the displacements of the balls are described by the system

$$\begin{aligned} x_1 &= l\cos\theta_1, \\ y_1 &= l\sin\theta_1, \\ x_2 &= l\cos\theta_1 + l\cos\theta_2, \\ y_2 &= l\sin\theta_1 + l\sin\theta_2. \end{aligned}$$

Now, differentiate with respect to time using Newton's dot notation for such derivatives:

$$\begin{aligned} \dot{x}_1 &= -l\sin\theta_1 \cdot \dot{\theta}_1, \\ \dot{y}_1 &= l\cos\theta_1 \cdot \dot{\theta}_1, \\ \dot{x}_2 &= -l\sin\theta_1 \cdot \dot{\theta}_1 - l\sin\theta_2 \cdot \dot{\theta}_2, \\ \dot{y}_2 &= l\cos\theta_1 \cdot \dot{\theta}_1 + l\cos\theta_2 \cdot \dot{\theta}_2. \end{aligned}$$

Since $v^2 = \dot{x}^2 + \dot{y}^2$, we get, after a bit of trigonometry,

$$\begin{aligned} v_1^2 &= l^2(\dot{\theta}_1)^2, \\ v_2^2 &= l^2((\dot{\theta}_1)^2 + \cos(\theta_1 - \theta_2)\dot{\theta}_1\dot{\theta}_2 + (\dot{\theta}_2)^2). \end{aligned}$$

So, the kinetic energy of the system is

$$K = \frac{1}{2}m(v_1^2 + v_2^2) = \frac{1}{2}ml^2(2(\dot{\theta}_1)^2 + \cos(\theta_1 - \theta_2)\dot{\theta}_1\dot{\theta}_2 + (\dot{\theta}_2)^2).$$

To rewrite this expression as a quadratic form determined by a symmetric matrix, we must split the matrix coefficient of $\dot{\theta}_1\dot{\theta}_2$ into two equal parts. In this way, we obtain:

$$K = \frac{1}{2}ml^2 \begin{bmatrix} \dot{\theta}_1 & \dot{\theta}_2 \end{bmatrix} \begin{bmatrix} 2 & \frac{1}{2}\cos(\theta_1 - \theta_2) \\ \frac{1}{2}\cos(\theta_1 - \theta_2) & 1 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}.$$

The potential energy of the system is purely gravitational. Relative to our reference point $P$, the first ball is at height $-x_1$ and the second at height $x_2$ where $x_1$ and $x_2$ are the $x$-displacements as defined in the beginning. So, the potential energy of the system is

$$V = -mgl(2\cos\theta_1 + \cos\theta_2).$$

Taking into account that $\cos\theta \approx 1 - \frac{\theta^2}{2}$ when $\theta$ is small, we approximate $V$ as

$$V \approx -mgl(3 + 2\theta_1^2 + \theta_2^2) = -3mgl - mgl \begin{bmatrix} \theta_1 & \theta_2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}.$$

A corresponding approximation for $K$ is

$$K = \frac{1}{2}ml^2 \begin{bmatrix} \dot{\theta}_1 & \dot{\theta}_2 \end{bmatrix} \begin{bmatrix} 2 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}.$$

Now we need to simultaneously diagonalize the (symmetric) matrices

$$K_1 := \begin{bmatrix} 2 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix} \qquad \text{and} \qquad K_2 := \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}.$$

Simultaneous diagonalization corresponds to a change of basis applied to $[\theta_1\theta_2]$ (the same change therefore must be applied $[\dot{\theta}_1\dot{\theta}_2]$). First let us make the second matrix into the identity by using

$$C = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & 1 \end{bmatrix}.$$

Then

$$C^T K_1 C = \begin{bmatrix} 1 & \frac{1}{2\sqrt{2}} \\ \frac{1}{2\sqrt{2}} & 1 \end{bmatrix} \qquad \text{and} \qquad C^T K_2 C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Then we need to find eigenvalues and eigenvectors of $C^T K_1 C$. Its characteristic equation is $(1-\lambda)^2 - \frac{1}{8}$ and its roots, i.e., eigenvalues of $CK_1C^T$, are $1 \pm \frac{1}{2\sqrt{2}}$. The corresponding (normalized to have length 1) eigenvectors are

$$v_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \qquad v_2 = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}.$$

Putting them next to each other, we obtain an orthogonal matrix

$$D := \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}.$$

Finally,

$$D^T C^T K_1 C D = \begin{bmatrix} 1 + \frac{1}{2\sqrt{2}} & 0 \\ 0 & 1 - \frac{1}{2\sqrt{2}} \end{bmatrix} \qquad \text{and} \qquad D^T C^T K_2 C D = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

So, our (approximations to) normal modes are $\frac{1}{\sqrt{2}}\theta_1 = \pm\theta_2$.

**Hilbert matrix problem.** (The matrix with elements $\frac{1}{i+j-1}$ is called the Hilbert matrix.)

Let us proceed by Gaussian elimination. Subtract the first row from the second row multiplied by 2, from the third row multiplied by 3 and from the fourth row multiplied by 4. We get

$$\begin{aligned} x_1 + \frac{1}{2}x_2 + \frac{1}{3}x_3 + \frac{1}{4}x_4 &= 1 \\ \frac{1}{6}x_2 + \frac{1}{6}x_3 + \frac{3}{20}x_4 &= 3 \\ \frac{1}{4}x_2 + \frac{4}{15}x_3 + \frac{1}{4}x_4 &= 8 \\ \frac{3}{10}x_2 + \frac{1}{3}x_3 + \frac{9}{28}x_4 &= 15. \end{aligned}$$

Multiply the second row by 6, the third by 4 and the last by 10 to get

$$\begin{aligned} x_1 + \frac{1}{2}x_2 + \frac{1}{3}x_3 + \frac{1}{4}x_4 &= 1 \\ x_2 + x_3 + \frac{9}{10}x_4 &= 18 \\ x_2 + \frac{16}{15}x_3 + x_4 &= 32 \\ 3x_2 + \frac{10}{3}x_3 + \frac{45}{14}x_4 &= 150. \end{aligned}$$

Subtract the second row from the third and the second row multiplied by 3 from the fourth row to get:

$$\begin{aligned} x_1 + \frac{1}{2}x_2 + \frac{1}{3}x_3 + \frac{1}{4}x_4 &= 1 \\ x_2 + x_3 + \frac{9}{10}x_4 &= 18 \\ \frac{1}{15}x_3 + \frac{}{10}x_4 &= 14 \\ \frac{1}{3}x_3 + \frac{18}{35}x_4 &= 196. \end{aligned}$$

Finally, subtract the third row multiplied by 5 from the fourth row. We get

$$\begin{aligned} x_1 + \frac{1}{2}x_2 + \frac{1}{3}x_3 + \frac{1}{4}x_4 &= 1 \\ x_2 + x_3 + \frac{9}{10}x_4 &= 18 \\ \frac{1}{15}x_3 + \frac{}{10}x_4 &= 14 \\ \frac{1}{70}x_4 &= 26. \end{aligned}$$

So, $x_4 = 1820$ and by backsubstitution we find $x_3 = -2520$, $x_2 = 900$, $x_1 = -64$.

**Answer:** $x_1 = -64$, $x_2 = 900$, $x_3 = -2520$, $x_4 = 1820$.