Solutions to homework # 3

II.16. (a) $f(z) = z^2 \sin z$. Since

$$\begin{aligned}\sin z &= \frac{e^{\mathrm{i}z} - e^{-\mathrm{i}z}}{2\mathrm{i}} = \frac{e^{\mathrm{i}(x+\mathrm{i}y)} - e^{-\mathrm{i}(x+\mathrm{i}y)}}{2\mathrm{i}} = \frac{e^{-y}(\cos x + \mathrm{i}\sin x) - e^{y}(\cos x - \mathrm{i}\sin x)}{2\mathrm{i}} \\ &= \frac{(e^{-y} + e^{y})\sin x}{2} + \mathrm{i}\frac{\cos x(e^{y} - e^{-y})}{2}\end{aligned}$$

and $z^2 = (x^2 - y^2) + 2\mathrm{i}xy$, we get $f(x,y) = u(x,y) + \mathrm{i}v(x,y)$ where

$$\begin{aligned}u(x,y) &= \frac{(x^2 - y^2)(e^{y} + e^{-y})\sin x}{2} - xy(e^{y} - e^{-y})\cos x \\ v(x,y) &= xy(e^{y} + e^{-y})\sin x + \frac{(x^2 - y^2)(e^{y} - e^{-y})\cos x}{2}.\end{aligned}$$

Then

$$\begin{aligned}\frac{\partial u}{\partial x} &= (2x\sin x + (x^2 - y^2)\cos x)\frac{e^{y} + e^{-y}}{2} - y(e^{y} - e^{-y})(\cos x - x\sin x) = \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} &= x\sin x(e^{y} + e^{-y} + y(e^{y} - e^{-y})) + \cos x((x^2 - y^2)\frac{e^{y} + e^{-y}}{2} - y(e^{y} - e^{-y})) = -\frac{\partial v}{\partial x}.\end{aligned}$$

(b) $f(z) = 1/(1+z)$. We have

$$\frac{1}{1+z} = \frac{1}{1+x+\mathrm{i}y} = \frac{1+x-\mathrm{i}y}{(1+x)^2 + y^2},$$

so $f(x,y) = u(x,y) + \mathrm{i}v(x,y)$ where

$$u(x,y) = \frac{1+x}{(1+x)^2 + y^2}, \qquad v(x,y) = -\frac{y}{(1+x)^2 + y^2}.$$

Hence

$$\begin{aligned}\frac{\partial u}{\partial x} &= \frac{y^2 - (1+x)^2}{((1+x)^2 + y^2)^2} = \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} &= \frac{-2(1+x)y}{((1+x)^2 + y^2)^2} = -\frac{\partial v}{\partial x}.\end{aligned}$$

II.17. Note that

$$\frac{\partial f}{\partial \bar{z}} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \bar{z}} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \bar{z}},$$

and

$$x = \frac{1}{2}(z + \bar{z}), \quad y = \frac{1}{2\mathrm{i}}(z - \bar{z}), \qquad \text{so} \quad \frac{\partial x}{\partial \bar{z}} = \frac{1}{2}, \quad \frac{\partial y}{\partial \bar{z}} = -\frac{1}{2\mathrm{i}}.$$

So,

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\frac{\partial f}{\partial x} - \frac{1}{2\mathrm{i}}\frac{\partial f}{\partial y} = \frac{1}{2}\left(\frac{\partial u}{\partial x} + \mathrm{i}\frac{\partial v}{\partial x}\right) - \frac{1}{2\mathrm{i}}\left(\frac{\partial u}{\partial y} + \mathrm{i}\frac{\partial v}{\partial y}\right) = \frac{1}{2}\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) + \frac{\mathrm{i}}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right).$$

The last expression is zero if and only if its real and imaginary parts are both zero, i.e.,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

II.18. To be the real (or imaginary) part of an analytic function, the function has to be harmonic, i.e., to satisfy Laplace's equation

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0.$$

Since

$$\frac{\partial^2 u_1}{\partial x^2} = 4 = \frac{\partial^2 u_1}{\partial y^2},$$

the function u_1 is not harmonic. but

$$\frac{\partial^2 u_2}{\partial x^2} = 2x = -\frac{\partial^2 u_2}{\partial y^2},$$

so u_2 is harmonic. The corresponding imaginary part $v_2(x, y)$ can be found from the Cauchy-Riemann equations:

$$\frac{\partial v_2}{\partial y} = \frac{\partial u_2}{\partial x} = x^2 - y^2,$$

so $v_2(x, y) = x^2 y - y^3/3 + g(x)$, and

$$2xy + g'(x) = \frac{\partial v_2}{\partial x} = -\frac{\partial u_2}{\partial y} = 2xy,$$

so $g(x) = c$ is constant. Thus,

$$f = u_2 + \mathrm{i}v_2 = \left(\frac{x^3}{3} - xy^2\right) + \mathrm{i}\left(x^2 y - \frac{y^3}{3} + c\right) = \frac{z^3}{3} + \mathrm{i}c.$$

II.21ab. (a) Using the Taylor series for $\cos z$, we get

$$z\cos z = z\left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots\right) = z - \frac{z^3}{2!} + \frac{z^5}{4!} - \cdots.$$

The Taylor series for cos converges everywhere, hence so does the obtained series for $z\cos z$.

(b) Since $(\ln(1+z))' = \frac{1}{1+z}$ and the latter is a geometric series, we have

$$(\ln(1+z))' = 1 - z + z^2 - z^3 + \cdots.$$

The radius of convergence of this series is 1. Integrating term by term and taking into account the fact $\ln 1 = 0$, we get

$$\ln(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \cdots .$$

Integration of a series preserves its radius of convergence, so it stays equal to 1.

II.22ab. (a) First determine the Taylor series of $\cos z$ centered at $z = 1$. Since $(\cos z)' = -\sin z$, $(\cos z)'' = -\cos z$, $(\cos z)''' = \sin z$, and $(\cos z)^{(4)} = \cos z$, after which the derivative sequence repeats itself, we get

$$\cos z = \cos 1 - \frac{\sin 1}{1!}(z-1) - \frac{\cos 1}{2!}(z-1)^2 + \frac{\sin 1}{3!}(z-1)^3 + \frac{\cos 1}{4!}(z-1)^4 - \cdots .$$

Therefore the Laurent series for $\frac{\cos z}{z-1}$ is

$$\frac{\cos z}{z-1} = \frac{\cos 1}{z-1} - \frac{\sin 1}{1!} - \frac{\cos 1}{2!}(z-1) + \frac{\sin 1}{3!}(z-1)^2 + \frac{\cos 1}{4!}(z-1)^3 - \cdots .$$

The Taylor series for $\cos z$ converges everywhere, hence the Laurent series for $\frac{\cos z}{z-1}$ converges everywhere except for $z = 1$.

(b) The Taylor series for $\sin z^2$ can be obtained from the Taylor series for $\sin z$ by substituting z^2 for z. Thus

$$\sin z^2 = z^2 - \frac{z^6}{3!} + \frac{z^{10}}{5!} - \frac{z^{14}}{7!} + \cdots .$$

The Taylor series for $\frac{\sin z^2}{z}$ is obtained by dividing through by z:

$$\frac{\sin z^2}{z} = z - \frac{z^5}{3!} + \frac{z^9}{5!} - \frac{z^{13}}{7!} + \cdots .$$

It converges everywhere since the series for $\sin z^2$ converges everywhere.

II.23ab. (a) The function $1/(1+z^2)$ is analytic everywhere except for its poles at $\pm\mathrm{i}$, hence it has two series centered at 0: a Taylor series for $|z| < 1$ and a Laurent series for $|z| > 1$. Multiplying the Taylor series for e^z by the Taylor series for $1/(1+z^2)$, we get the following Taylor series for $e^z/(1+z^2)$ for $|z| < 1$:

$$\begin{aligned}\frac{e^z}{1+z^2} &= \left(1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots\right)\left(1 - z^2 + z^4 - z^6 + \cdots\right) \\ &= \sum_{n=0}^{\infty} \sum_{n-2k \geq 0} \frac{z^{n-2k}}{(n-2k)!}(-1)^k z^{2k} = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^k}{(n-2k)!} \right) z^n .\end{aligned}$$

For $|z| > 1$, we have:

$$\begin{aligned}\frac{e^z}{1+z^2} &= \frac{e^z}{z^2} \cdot \frac{1}{1+1/z^2} = \left(\frac{1}{z^2} + \frac{1}{1!z} + \frac{1}{2!} + \frac{z}{3!} + \cdots\right)\left(1 - \frac{1}{z^2} + \frac{1}{z^4} - \frac{1}{z^6} + \cdots\right) \\ &= \sum_{n=-\infty}^{\infty} \sum_{\substack{n+2k+2 \geq 0 \\ k \geq 0}} \frac{z^{n+2k}}{(n+2k+2)!} \frac{(-1)^k}{z^{2k}} = \sum_{n=-\infty}^{\infty} \left(\sum_{k=\max\{0, \lfloor -\frac{n}{2} \rfloor - 1\}}^{\infty} \frac{(-1)^k}{(n+2k+2)!} \right) z^n .\end{aligned}$$

Since the Taylor series for e^z and for e^{-z} converge everywhere, multiplication of an arbitrary series by e^z does not change its radius of convergence. Therefore, the first series converges for $|z| < 1$ and the second for $|z| > 1$.

(b) The function has two poles, at $z = \mathrm{i}$ and $z = -\mathrm{i}$. Hence there are two Laurent series centered at $z = \mathrm{i}$, one valid in the region $0 < |z-\mathrm{i}| < 2$ and the other in the region $2 < |z-\mathrm{i}|$.

For $0 < |z - \mathrm{i}| < 2$, we have

$$\begin{aligned}
\frac{1}{z^2+1} &= \frac{1}{(z+\mathrm{i})(z-\mathrm{i})} = \frac{1}{2\mathrm{i}(z-\mathrm{i})}\frac{1}{1+\frac{z-i}{2\mathrm{i}}} = \frac{1}{2\mathrm{i}(z-\mathrm{i})}\left(1 - \frac{z-\mathrm{i}}{2\mathrm{i}} + \left(\frac{z-\mathrm{i}}{2\mathrm{i}}\right)^2 - \cdots\right) \\
&= \frac{1}{2\mathrm{i}(z-\mathrm{i})} - \frac{1}{(2\mathrm{i})^2} + \frac{z-\mathrm{i}}{(2\mathrm{i})^3} - \frac{(z-i)^2}{(2\mathrm{i})^4} + \cdots = \sum_{n=-1}^{\infty} (-1)^{n+1}\frac{(z-\mathrm{i})^n}{(2\mathrm{i})^{n+2}}.
\end{aligned}$$

The geometric series $1/(1+\frac{z-\mathrm{i}}{2\mathrm{i}})$, and hence the obtained Laurent series, converge for $|z-\mathrm{i}| < |2\mathrm{i}| = 2$. For $2 < |z - \mathrm{i}|$, we have to expand $1/(z + \mathrm{i})$ differently:

$$\begin{aligned}
\frac{1}{z^2+1} &= \frac{1}{(z+\mathrm{i})(z-\mathrm{i})} = \frac{1}{(z-\mathrm{i})^2}\frac{1}{1+\frac{2\mathrm{i}}{z-\mathrm{i}}} = \frac{1}{(z-\mathrm{i})^2}\left(1 - \frac{2\mathrm{i}}{z-\mathrm{i}} + \left(\frac{2\mathrm{i}}{z-\mathrm{i}}\right)^2 - \cdots\right) \\
&= \frac{1}{(z-\mathrm{i})^2} - \frac{2\mathrm{i}}{(z-\mathrm{i})^3} + \frac{(2\mathrm{i})^2}{(z-\mathrm{i})^4} - \frac{(2\mathrm{i})^3}{(z-\mathrm{i})^5} + \cdots = \sum_{n=2}^{\infty}(-1)^n\frac{(2\mathrm{i})^{n-2}}{(z-\mathrm{i})^n}.
\end{aligned}$$

The geometric series $1/(1 + \frac{2\mathrm{i}}{(z-\mathrm{i})})$ and the Laurent series converge for $|z - \mathrm{i}| > |2\mathrm{i}| = 2$.