

**II.27.**

$$(a) \quad \operatorname{Res} f(i) = \lim_{z \rightarrow i} (z - i)f(z) = \lim_{z \rightarrow i} (z - i) \frac{z - 2}{(z - i)(z + i)} = \lim_{z \rightarrow i} \frac{z - 2}{z + i} = \frac{i - 2}{2i}.$$

(b) Since  $e^{1/z-1} = e^{-1}e^{1/z}$ , we get

$$e^{1/z-1} = e^{-1} \left( 1 + \frac{1}{1!z} + \frac{1}{2!z^2} + \cdots \right).$$

The coefficient  $c_{-1}$  of  $1/z$  in this Laurent series, i.e.,  $\operatorname{Res} f(0)$ , is equal to  $e^{-1}$ .

(c) Using the Taylor expansion of  $\sin z$  at the origin, we get:

$$\frac{\sin z}{z^2} = \frac{1}{z^2} \left( z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots \right) = \frac{1}{z} - \frac{z}{3!} + \frac{z^3}{5!} - \cdots$$

The coefficient  $c_{-1}$  of  $1/z$  in this Laurent series is equal to 1, and this is nothing but  $\operatorname{Res} f(0)$ .

$$(d) \quad \operatorname{Res} f\left(\frac{\pi}{6}\right) = \lim_{z \rightarrow \frac{\pi}{6}} \frac{\cos z}{\frac{d}{dz}(1/2 - \sin z)} = \lim_{z \rightarrow \frac{\pi}{6}} \frac{\cos z}{(-\cos z)} = -1.$$

**II.28.** (a) The circle  $|z| = 2$  encloses the only singularity of the function  $f(z) := \frac{\cos z}{z}$ , namely, its pole of order 1 at the origin. Since

$$\operatorname{Res} f(0) = \lim_{z \rightarrow 0} z f(z) = \lim_{z \rightarrow 0} \cos z = 1,$$

we conclude

$$\oint_C \frac{\cos z}{z} dz = 2\pi i \operatorname{Res} f(0) = 2\pi i.$$

(b) Analogously to (a) above, the square of side length 4 centered at the origin encloses the only singularity of the function  $f(z) := \frac{\sinh z}{z-1}$ , which is a simple pole. Therefore

$$\operatorname{Res} f(1) = \lim_{z \rightarrow 1} (z - 1)f(z) = \lim_{z \rightarrow 1} \sinh z = \sinh 1,$$

and

$$\oint_C \frac{\sinh z}{z-1} dz = 2\pi i \operatorname{Res} f(1) = 2\pi i \sinh 1.$$

(c) The function  $\frac{z-1}{z+2}$  is analytic everywhere inside the circle of radius 1 centered at the origin, hence the integral in question is equal to zero.

(d) The function  $f(z) := \frac{z}{4z^2+1}$  has two simple poles at  $z = \pm \frac{i}{2}$  and is analytic elsewhere. The square  $C$  encloses only one of the poles, viz.,  $z = \frac{i}{2}$ . Hence

$$\oint_C \frac{z}{4z^2+1} dz = 2\pi i \operatorname{Res} f\left(\frac{i}{2}\right) = 2\pi i \lim_{z \rightarrow i/2} \left(z - \frac{i}{2}\right) f(z) = 2\pi i \lim_{z \rightarrow i/2} \frac{z}{4(z + i/2)} = \frac{\pi i}{4}.$$

**II.30.** (a) Given a large positive number  $R$ , consider the contour  $\Gamma_R$  consisting of the interval  $[-R, R]$  of the real axis traversed from left to right together with the half-circle  $C_R$   $Re^{i\phi}$ , where  $\phi$  runs from 0 to  $\pi$ . Since

$$\left| \oint_{C_R} \frac{dz}{z^2+2} \right| \leq \frac{\operatorname{Arclength}(C_R)}{R^2-2} = \frac{\pi R}{R^2-2} \rightarrow 0 \quad \text{as } R \rightarrow \infty,$$

we see that

$$\lim_{R \rightarrow \infty} \oint_{\Gamma_R} \frac{dz}{z^2+2} = \int_{-\infty}^{\infty} \frac{dx}{x^2+2}.$$

For all large  $R$ , the contour  $\Gamma_R$  encloses exactly one simple pole of the function  $f(z) := \frac{1}{z^2+2}$  at  $z = \sqrt{2}i$ . Hence

$$\int_{-\infty}^{\infty} \frac{dx}{x^2+2} = 2\pi i \operatorname{Res} f(\sqrt{2}i) = 2\pi i \lim_{z \rightarrow \sqrt{2}i} (z - \sqrt{2}i) f(z) = 2\pi i \lim_{z \rightarrow \sqrt{2}i} \frac{1}{z + \sqrt{2}i} = \frac{2\pi i}{2\sqrt{2}i} = \frac{\pi}{\sqrt{2}}.$$

(b) To find the principal value of the given integral, take a large  $R > 0$  and a small  $\varepsilon > 0$  and consider the contour  $\Gamma_{R,\varepsilon}$  consisting of the segments  $[-R, -1 - \varepsilon]$ , the half circle  $C_\varepsilon: -1 + \varepsilon e^{i\phi}$ , with  $\phi$  running from  $\pi$  to 0, the segment  $[-1 + \varepsilon, R]$  and the half-circle  $C_R: Re^{i\phi}$ , with  $\phi$  running from 0 to  $\pi$ . Then

$$\left| \oint_{C_R} \frac{z dz}{z^3+1} \right| \leq \frac{\operatorname{Arclength}(C_R)R}{R^3-1} = \frac{\pi R^2}{R^3-1} \rightarrow 0 \quad \text{as } R \rightarrow \infty$$

and

$$\oint_{C_\varepsilon} \frac{z dz}{z^3+1} = \int_\pi^0 \frac{(-1 + \varepsilon e^{i\phi}) \varepsilon i e^{i\phi} d\phi}{(-1 + \varepsilon e^{i\phi})^3 + 1} = \int_\pi^0 \frac{(-1 + \varepsilon e^{i\phi}) i e^{i\phi} d\phi}{3e^{i\phi} - 3\varepsilon e^{2i\phi} + \varepsilon^2 e^{3i\phi}} \rightarrow \int_\pi^0 \frac{-i d\phi}{3} = \frac{\pi i}{3} \quad \text{as } \varepsilon \rightarrow 0.$$

For all sufficiently large  $R$  and sufficiently small  $\varepsilon$ , the contour  $\Gamma_{R,\varepsilon}$  encloses one of the poles of the integrand, namely, the point  $z = -e^{-2\pi i/3}$ . Hence, for large  $R$  and small  $\varepsilon$ ,

$$\oint_{\Gamma_{R,\varepsilon}} \frac{z dz}{z^3+1} = 2\pi i \lim_{z \rightarrow -e^{-2\pi i/3}} \frac{z}{(z+1)(z+e^{2\pi i/3})} = \frac{2\pi i (-e^{-\pi i/3})}{(2i)^2 \sin \frac{\pi}{3} \sin \frac{2\pi}{3}} = \frac{\pi}{\sqrt{3}} + \frac{\pi i}{3}.$$

Thus

$$P \int_{-\infty}^{\infty} \frac{x dx}{x^3+1} = \lim_{R \rightarrow \infty, \varepsilon \rightarrow 0} \oint_{\Gamma_{R,\varepsilon}} \frac{z dz}{z^3+1} - \frac{\pi i}{3} = \frac{\pi}{\sqrt{3}}.$$